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A refinement of the Strichartz inequality with applications to the linear and nonlinear wave equations

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Mi gratitud hacia Ana y Keith va más allá de unas frases bien elegidas.

Resumen y conclusiones

En el primer capítulo de esta memoria se probará una mejora de la desigualdad clásica de Strichartz para la ecuación lineal de ondas. La desigualdad de Strichartz permite acotar soluciones de la ecuación lineal de ondas en el espacio $L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})$ en términos de la norma $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ de los datos iniciales. Nuestra mejora permitirá acotar a partir de un espacio más amplio de datos iniciales en el que podremos asegurar, entre otras cosas, que si la norma de la solución es grande, entonces hay una conjunto en el que el dato inicial se concentra en frecuencia. La estimación se puede entender como un teorema de restricción al cono de la transformada de Fourier en su versión dual.

La prueba hace uso de un teorema bilineal de restricción para el cono probado por Tao. Este considera pares de funciones soportadas en frecuencia en subconjuntos del cono que tienen cierta separación angular entre sí y viven en distintos anillos diádicos. Este resultado mejora al establecido previamente por Wolff, entre otros motivos porque permite considerar distintos anillos diádicos. Esto será clave para conseguir la mejora en la desigualdad. Para poder utilizar el teorema bilineal de Tao necesitaremos ciertas descomposiciones en el espacio de frecuencia siguiendo los argumentos de Tao, Vargas y Vega. La dificultad adicional en nuestro caso se debe a que en dimensiónes $d \neq 3$, tendremos que considerar propiedades de ortogonalidad en entornos del cono en L^p con $p \neq 2$, que es una cuestión antigua y todavía abierta. Conseguiremos superar esta dificultad gracias a consideraciones geométricas que nos permitirán utilizar una generalización de un lema de ortogonalidad probado por Tao, Vargas y Vega, y una descomposición atómica debida a Keel y Tao.

Las primeras mejoras de la desigualdad de Strichartz fueron conseguidas para la ecuación de Schrödinger en dimensión $d = 2$ por Bourgain, y Moyua, Vargas y Vega. Estos resultados fueron posteriormente extendidos a todas las dimensiones, y generalizados para diferentes ecuaciones dispersivas como la ecuación de Schrödinger no elíptica y de órden superior, para Klein–Gordon o para la ecuación de Airy.

En el segundo capítulo conseguiremos una descomposición en perfiles para la ecuación lineal de ondas con dato inicial en $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ con $d \geq 2$. Esta nos permitirá caracterizar la pérdida de compacidad del operador de ondas. La descomposición en perfiles permite escribir cualquier secuencia de soluciones en términos de una familia de soluciones, llamadas perfiles, que se van transformando mediante dilataciones, traslaciones, y transformadas de Lorentz. También podremos asegurar que estas transformaciones son asintóticamente ortogonales, en el sentido de que en el límite los perfiles transformados no interaccionan unos con otros. Además de estas transformaciones tendremos un término de resto que se hará tan pequeño como queramos según añadamos más perfiles. La prueba de la descomposición en perfiles se basa en los trabajos pioneros de Gérard y Bahouri para la ecuación de ondas con dato inicial en $\dot{H}^1 \times L^2(\mathbb{R}^3)$, y Merle y Vega para la ecuación de Schrödinger con dato inicial en $L^2(\mathbb{R}^2)$. En los últimos años numerosos autores han obtenido la descomposición en perfiles para diversas ecuaciones dispersivas. A diferencia de dichas descomposiciones, en nuestro caso tendremos que considerar el defecto de compacidad causado por la transformada de Lorentz. El elemento clave en la prueba será la mejora de la desigualdad de Strichartz del primer capítulo.

Como aplicación directa de esta descomposición podremos probar que la desigualdad de Strichartz tiene maximizantes, en el sentido de que hay funciones que alcanzan la constante de la desigualdad. Probar la existencia de maximizantes y describirlos es un área de gran actividad recientemente.

En el último capítulo trabajaremos con la ecuación no lineal y crítica de ondas con regularidad $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ y $d \geq 3$. Probaremos que en el caso de que la ecuación tenga explosión de tipo II (es decir que su norma de Strichartz se haga infinito pero la norma $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ esté acotada) entonces la solución se concentra en ciertos rectángulos cuyo lado mayor es proporcional al inverso de la distancia al tiempo de explosión. Esto podría ser útil para probar que la ecuación no puede tener este tipo de explosión. En efecto, este tipo de argumento ha sido muy útil recientemente para el caso de la ecuación de Schrödinger y para la ecuación de ondas con dato inicial en $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ con $s \geq 1$. Destacamos que el nuestro es el primer resultado de este tipo para le ecuación de ondas no lineal con tan poca regularidad en el dato. En efecto, los únicos resultados obtenidos hasta la fecha para la ecuación no lineal de ondas necesitan regularidad al menos $\dot{H}^1 \times L^2$. Nuevamente, la mejora de la desigualdad de Strichartz jugará un papel clave.

Summary and results

In the first chapter of the thesis a refinement of the classical Strichartz inequality for the wave equation will be proven. The Strichartz inequality allows one to bound solutions of the linear wave equation in the space $L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})$ in terms of the norm $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ of the initial data. Our refinement will allows us to bound in terms of a larger space of initial data in which, among other things, we can ensure that if the norm of the solution is big, then there is a set in which the initial data concentrates in frequency. The inequality can be thought as a dual version of the Fourier restriction inequality for the cone.

The proof uses a bilinear restriction theorem for the cone proven by Tao. It considers pairs of functions frequency supported in subsets of the cone with angular distance between and in different dyadic annuli. This result improves the one previously proven by Wolff, among other things, because it permits to handle different dyadic annuli. This will be essential to improve the Strichartz inequality. In order to use the bilinear theorem of Tao we need to decompose the frequency space using similar arguments to those of Tao, Vargas and Vega. The additional problem in our case is that when $d \neq 3$, we are lead to consider orthogonality properties of thickened pieces of the cone in L^p with $p \neq 2$, which is a deep and largely unanswered question. We sidestep the problem by strengthening the standard lemma which proves that the norm on the right hand side of our refinement is smaller than the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ norm. This is achieved using an atomic decomposition of L^p due to Keel and Tao.

The first refinements of the Strichartz inequality were proven for the Schrödinger equation with $d = 2$ by Bourgain, and Moyua, Vargas and Vega. These results were then extended to all dimensions, and generalized to different dispersive equations like the nonelliptic and higher order Schrödinger equations, the Klein–Gordon equation or the Airy equation.

In the second chapter we obtain a profile decomposition for the linear wave equation with initial data in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ and $d \geq 2$. This allows us to characterize the defect of compactness of the wave operator. In addition, the profile decomposition allows us to write any sequence of solutions as a sum of transformations of a family of solution called profiles. These transformations are dilations, space–time translations and Lorentz transformations; and are asymptotically orthogonal, in the sense that the transformed profiles do not interact with each other in the limit. Together with

these transformations we need to add a remainder term, which will be as small as we want if we add enough profiles. The proof of the profile decomposition is based on the pioneering works of Gérard and Bahouri for the wave equation with initial data in $\dot{H}^1 \times L^2(\mathbb{R}^3)$; and Merle and Vega for the Schrödinger equation with initial data in $L^2(\mathbb{R}^2)$. Recently many authors have obtained a profile decomposition for different dispersive equations. Unlike these decompositions, in our case we have to deal with the defect of compactness caused by the Lorentz transformations. The key ingredient in the proof is the Strichartz refinement of the first chapter.

As a direct application of the profile decomposition we prove that the Strichartz refinement has maximizers, in the sense that we can find functions which attain the constant of the inequality. Proving the existence of maximizers and characterizing them is an area of intense activity lately.

In the last chapter we work with the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ critical nonlinear wave equation with $d \geq 3$. We prove that if there is a blow-up of type II (that is, the Strichartz norm is infinite but the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ norm is bounded), then the solution concentrates in some rectangles whose larger side is proportional to the inverse of the distance to the time of the blow-up. This could be useful to prohibit the existence of type II blow-up solutions. These kind of arguments were very useful for the Schrödinger equation and the wave equation with initial data in $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ with $s \geq 1$. This is the first result of its kind for the nonlinear wave equation with such little regularity. Indeed, the only results obtained so far for the nonlinear wave equation require the data to be contained in $\dot{H}^1 \times L^2$. Again, the Strichartz refinement plays a key role.

Contents

Introduction

This thesis is concerned with a refinement of a restriction theorem for the Fourier transform and its applications to the linear and nonlinear wave equations.

In the first chapter, we prove a refinement of a classical inequality proven by Strichartz in 1977. In the second chapter, we show a profile decomposition for the linear wave equation. A direct consequence of this is that there exists a maximizer for the inequality proven by Strichartz. In the last chapter we prove a result about blow-up solutions of the nonlinear $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ critical wave equation. The key ingredient of the proofs of the results in the second and third chapters is the Strichartz refinement.

0.1. The Strichartz refinement

We present a brief overview of the restriction theory focusing on the refinement, introducing the basics of the subject.

What is a Strichartz refinement?

Given a surface S in $\mathbb{R}^{d+1}$ endowed with its canonical measure $d\sigma$, Stein proposed the study of the operator which assigns to every function the restriction to S of its Fourier transform. In particular, determine the values of p and q for which the restriction of the Fourier transform of every function $g \in L^p(\mathbb{R}^{d+1})$ makes sense in $L^q(S, d\sigma)$. The first step to understand this problem is to know where the Fourier transform of a function of L^p lies. From the Hausdorff–Young inequality

$$\|\widehat{g}\|_{L^{p'}(\mathbb{R}^{d+1})} \leq \|g\|_{L^p(\mathbb{R}^{d+1})} \quad \text{for } \ 1 \leq p \leq 2,$$

we can deduce that, for $1 \leq p \leq 2$, a function in $L^p(\mathbb{R}^{d+1})$ is in $L^{p'}(\mathbb{R}^{d+1})$. As these spaces are defined almost everywhere, we hope that $\widehat{g}$ is in a good subset of $L^{p'}(\mathbb{R}^{d+1})$ where the restriction can make sense. For example, the Fourier transform of a function in $L^1(\mathbb{R}^{d+1})$ is a continous bounded function, which permits a meaningful restriction. On the other hand, the Fourier transform is a bijection from $L^2(\mathbb{R}^{d+1})$ to $L^2(\mathbb{R}^{d+1})$, so the restriction can not make sense for functions in $L^2(\mathbb{R}^{d+1})$. In between $p = 1$ and $p = 2$, Stein conjectured, for the case of the sphere $\mathbb{S}^d := \{\xi \in \mathbb{R}^{d+1} : \ |\xi|^2 = 1\}$, that

$$\|\widehat{g}\|_{L^q(\mathbb{S}^d, d\sigma)} \leq C\|g\|_{L^p(\mathbb{R}^{d+1})} \quad \text{for every } \ p < 2\frac{d+1}{d+2} \ \text{ with } \ p' \geq \frac{d+2}{d}q. \tag{0.1}$$

It turns out that the Gaussian curvature of the surface plays a key role in such questions. Indeed, the only possible restriction estimates for the hyperplane are those with $p = 1$. It was conjectured that the estimate (0.1) should hold for the case of compact subsets of the paraboloid $P := \{(\underline{\xi}, \xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R} : \ |\underline{\xi}|^2 = \xi_{d+1}\}$ with the same range. For the cone

$$\mathcal{C} = \{(\underline{\xi}, \xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R} : \ |\underline{\xi}| = |\xi_{d+1}|\},$$

it is expected that for compact subsets of $\mathcal{C} \setminus \{0\}$ the restriction estimate holds for p and q in the range conjectured for the sphere in $\mathbb{R}^d$.

Conjecture 0.1. *For every $\mathcal{K} \subset \mathcal{C}\backslash\{0\}$ compact and for every $p < 2\frac{d}{d+1}$ with $p' \geq \frac{d+1}{d-1}q$,*

$$\|\widehat{g}\|_{L^q(\mathcal{K},d\sigma)} \leq C\|g\|_{L^p(\mathbb{R}^{d+1})}. \tag{0.2}$$

For the case of the cone we take the measure $d\sigma(\xi)$ as the pullback of the d dimensional measure $\frac{d\underline{\xi}}{|\underline{\xi}|}$ under the projection to the d first coordinates $(\underline{\xi}, \xi_{d+1}) \to \underline{\xi}$.

The conjecture for the sphere and paraboloid with $d = 1$ was resolved by Fefferman [30] and Zygmund [85] (see also [20]), and the conical case with $d = 2, 3$ was resolved by Barceló [2] and Wolff [84].

The conjecture for the sphere is open for dimensions $d \geq 2$, while for the cone it is open for dimensions $d \geq 4$. Nevertheless a lot of partial results have been achieved. One of the first improvements with respect to the trivial case, $p = 1$, for the sphere was proven by Tomas [81];

$$\|\widehat{g}\|_{L^2(\mathbb{S}^d,d\sigma)} \leq C\|g\|_{L^p(\mathbb{R}^{d+1})} \quad \text{for every} \ \ p < 2\frac{d+2}{d+4}, \tag{0.3}$$

and Stein [72] improved this to the endpoint $p = 2\frac{d+2}{d+4}$.

Strichartz [73] then proved the analogous results for the case of the paraboloid and the cone. We state now the result for the cone.

Theorem 0.1. [73]

$$\|\widehat{g}\|_{L^2(\mathcal{C},d\sigma)} \leq C\|g\|_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^{d+1})}. \tag{0.4}$$

As the cone is not compact, and due to its invariance under certain groups of dilations, the exponent $p = 2\frac{d+1}{d+3}$ cannot be replaced by any another number in (0.4). By trivial considerations from (0.4) we obtain

$$\|\widehat{g}\|_{L^2(\mathcal{K},d\sigma)} \leq C\|g\|_{L^p(\mathbb{R}^{d+1})} \quad \text{for all} \quad p < 2\frac{d+1}{d+3} \quad \text{and every compact} \quad \mathcal{K} \subset \mathcal{C} \setminus \{0\}.$$

The proof of these theorems relies strongly on the fact that $q = 2$. The way to reach an estimate with $q < 2$ in higher dimensions was not discovered until the nineties by Bourgain [7]. After that there have been improvements by Wolff [82], Moyua, Vargas and Vega [59], [60], Tao, Vargas and Vega [76], Tao and Vargas [77], Tao [75] and Bourgain and Guth [12] for the case of the paraboloid and the sphere; and Bourgain [9], Tao and Vargas [77] and Wolff [84] for the case of the cone. One of the main tools in these improvements was the bilinear point of view. The dual version of the cone restriction conjecture (0.2) takes the form

$$\|\widehat{gd\sigma}\|_{L^{p'}} \leq C\|g\|_{L^{q'}(\mathcal{K},d\sigma)}, \quad \text{for every } \; p' > 2\frac{d}{d-1} \; \text{ and } \; q' \geq \Big(\frac{(d-1)p'}{d+1}\Big)'.$$

The bilinear estimates take the form

$$\|\widehat{g_1 d\sigma}\widehat{g_2 d\sigma}\|_{L^r} \leq C\|g_1\|_{L^2(S,d\sigma)}\|g_2\|_{L^2(S,d\sigma)},$$

where the supports of g_1 and g_2 are separated and the constant C depends on this distance. The sharp exponent for this inequality is $r = \frac{d+3}{d+1}$ for cones and paraboloids. Tao [74] proved the sharp exponent for the cone. In the paraboloid case, Tao [75] was able to prove the best known estimate which is sharp except for the endpoint (see previous results by Bourgain [9], Tao, Vargas and Vega [76], and Wolff [84]). It turns out that from these bilinear estimates it is possible to deduce linear ones, as was proven by Tao, Vargas and Vega [76].

A multilinear estimate was proven by Bennett, Carbery and Tao [5]. The method to use multilinear estimates to deduce the linear ones was unclear for some time. It was achieved very recently by Bourgain and Guth [12] and currently the best known linear restriction estimate is obtained in this way.

The dual version of (0.4) can be stated as

$$\|\widehat{gd\sigma}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\|g\|_{L^2(\mathcal{C},d\sigma)}. \tag{0.5}$$

For this value of $p' = 2\frac{d+1}{d-1}$, the space $L^2(\mathcal{C}, d\sigma)$ is sharp. However, it is possible to refine the estimate, replacing L^2 by another space X such that $L^2(\mathcal{C}, d\sigma) \subset X$. Before describing the space of functions X, we motivate its definition through the linear wave equation. The estimate (0.5) and its refinement are very useful in the context of the wave equation.

Connection with the wave equation

Let $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ denote the homogeneous Sobolev space with half a derivative in $L^2(\mathbb{R}^d)$. The linear wave equation in $\mathbb{R}^{d+1}$

$$\begin{cases} u_{tt} = \Delta u \\ u(0) = u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d), \; u_t(0) = u_1 \in \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d), \end{cases} \tag{0.6}$$

has a solution which can be written as

$$S[u_0, u_1](t) = \frac{1}{2}\Big(e^{it\sqrt{-\Delta}}u_0 + \frac{1}{i}\frac{e^{it\sqrt{-\Delta}}u_1}{\sqrt{-\Delta}}\Big) + \frac{1}{2}\Big(e^{-it\sqrt{-\Delta}}u_0 - \frac{1}{i}\frac{e^{-it\sqrt{-\Delta}}u_1}{\sqrt{-\Delta}}\Big),$$

where

$$e^{\pm it\sqrt{-\Delta}}f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{i(x\cdot\xi\pm t|\xi|)}d\xi,$$

and $\widehat{\sqrt{-\Delta}f}(\xi)=|\xi|\widehat{f}(\xi)$. If we set $g(\xi,|\xi|)=|\xi|\widehat{f}(\xi)$ and $g(\xi,-|\xi|)=0$, this operator can be written in terms of the Fourier transform of a function supported in the upper cone, i.e $\widehat{gd\sigma}(x,t)=e^{it\sqrt{-\Delta}}f(x)$, so that restriction estimates for the cone become relevant. Similarly for $e^{-it\sqrt{-\Delta}}f(x)$.

Notice that the Strichartz estimate (0.5) can be written as

$$\|e^{\pm it\sqrt{-\Delta}}f\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\leq C\|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}.$$

In terms of the solution of the wave equation, this estimate implies

$$\|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\leq C\Big(\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}+\|u_1\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}\Big). \tag{0.7}$$

To explain our refinement we begin with a simpler version in terms of a Besov norm

$$;\|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\leq C\Big(\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^d)}+\|u_1\|_{\dot{B}^{-\frac{1}{2}}_{2,q}(\mathbb{R}^d)}\Big), \tag{0.8}$$

where $q=2\frac{d+1}{d-1}$ for $d\geq 3$, and $q=3$ for $d=2$. Here $\dot{B}^s_{2,q}$ is defined by

$$\|f\|_{\dot{B}^s_{2,q}}=(\sum_k 2^{ksq}\|P_kf\|_2^q)^{\frac{1}{q}},$$

where $\widehat{P_kg}=\chi_{\mathcal{A}_k}\widehat{g}$ and $\mathcal{A}_k=\{\xi\in\mathbb{R}^d;\ \ 2^k\leq|\xi|\leq 2^{k+1}\}$. The Strichartz estimate (0.7) follows from (0.8) by the embedding $\ell^2\hookrightarrow\ell^q$.

The main advantage of this refinement is that we can take a supremum without loss of regularity;

$$\|f\|_{\dot{B}^s_{2,q}}\leq\Big(\sup_k 2^{ks(q-2)}\|P_kf\|_2^{q-2}\Big)^{\frac{1}{q}}\|f\|_{\dot{H}^s}^{\frac{2}{q}}.$$

We can refine the Strichartz inequality further, in such a way that the supremum will be in some smaller norm than the L^2 norm. This will be fundamental for our applications. Let $M=\{w_m\}_m\subset\mathbb{S}^{d-1}$ be a maximally 2^{-j}-separated set, and define the sectors $\tau_m^{j,k}$ by

$$\tau_m^{j,k}:=\Big\{\xi\in\mathcal{A}_k:\big|\frac{\xi}{|\xi|}-w_m\big|\leq\big|\frac{\xi}{|\xi|}-w_{m'}\big|\ \text{ for every } w_{m'}\in M,\ m'\neq m\Big\}.$$

Note that $\big|\dfrac{\xi}{|\xi|}-w_m\big|\leq 2^{-j}$ for all $\xi\in\tau_m^{j,k}$. We also set $\widehat{P_kf_m^j}=\chi_{\tau_m^{j,k}}\widehat{f}$. We can now state our Strichartz refinement.

Theorem 1.1. *There exist* $p<2$ *and* $\theta_0>0$ *such that for all* $0\le\theta\le\theta_0$[1],

$$\|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\le C\Big(\sup_{j,k,m}2^{k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k(u_0)}_m^j\|_p^\theta\|u_0\|^{1-\theta}_{B^{\frac12}_{2,q(1-\theta)}}+\sup_{j,k,m}2^{-k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k(u_1)}_m^j\|_p^\theta\|u_1\|^{1-\theta}_{B^{-\frac12}_{2,q(1-\theta)}}\Big).$$

for $q=2\frac{d+1}{d-1}$ *if* $d\ge 3$, *and* $q=3$ *for* $d=2$.

A refinement of this type was obtained with $d=2$ for the Schrödinger equation by Bourgain [8] and generalized and improved by Moyua, Vargas and Vega [59], [60]. Carles and Keraani [14] obtained the result for dimension $d=1$, and by Bégout and Vargas [3] for dimensions $d\ge 3$. In these cases, the sets $\tau_m^{j,k}$ are replaced by squares, and the Besov norm by the L^2 norm.

Also Rogers and Vargas [67] proved a refinement of the Strichartz inequality for the nonelliptic Schrödinger equation. Chae, Hong and Lee [16] obtained a similar result for higher order Schrödinger equations, Shao [69] for the Airy equation, and Killip, Stovall and Visan [48] for the Klein–Gordon equation.

Outline of the proof.

The proof of this refinement relies on two key points:

How do we get the L^p norm with $p<2$ on the right hand side?

$$\|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\le C\Big(\sup_{j,k,m}2^{k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k(u_0)}_m^j\|_p^\theta\|u_0\|^{1-\theta}_{B^{\frac12}_{2,q(1-\theta)}}+\sup_{j,k,m}2^{-k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k(u_1)}_m^j\|_p^\theta\|u_1\|^{1-\theta}_{B^{-\frac12}_{2,q(1-\theta)}}\Big).$$

How do we improve from ℓ^2 to ℓ^q with $q>2$?

$$\|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\le C\Big(\sup_{j,k,m}2^{k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k(u_0)}_m^j\|_p^\theta\|u_0\|^{1-\theta}_{B^{\frac12}_{2,q(1-\theta)}}+\sup_{j,k,m}2^{-k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k(u_1)}_m^j\|_p^\theta\|u_1\|^{1-\theta}_{B^{-\frac12}_{2,q(1-\theta)}}\Big).$$

As for the first question, the answer is the bilinear estimates, while for the second the answer is the high-low frequency interaction in the bilinear estimates. We explain now the details emphasizing on the difficulties involved.

The bilinear method. As we already mentioned, the bilinear estimates assume some separation in the Fourier supports of the functions, and give some improvement in the

[1] See Theorem 1.2 and Theorem 1.3.

range of p and q with respect to the linear case. For the case of the cone, we need angular distance, in the sense that for every $\xi_1 \in \operatorname{supp} \widehat{f}_1$ and $\xi_2 \in \operatorname{supp} \widehat{f}_2$, we have

$$\Big| \frac{\xi_1}{|\xi_1|} - \frac{\xi_2}{|\xi_2|} \Big| \sim 1.$$

Under this condition, assuming that $\operatorname{supp} \widehat{f}_1, \operatorname{supp} \widehat{f}_2 \subset \mathcal{A}_0$, the bilinear estimates take the form

$$\|e^{it\sqrt{-\Delta}} f_1 \; e^{it\sqrt{-\Delta}} f_2\|_{L^q(\mathbb{R}^{d+1})} \leq C \|f_1\|_{L^2(\mathbb{R}^d)} \|f_2\|_{L^2(\mathbb{R}^d)}.$$

Notice that by the Cauchy–Schwarz inequality the linear estimate

$$\|e^{it\sqrt{-\Delta}} f\|_{L^{2q}(\mathbb{R}^{d+1})} \leq \, C \, \|f\|_{L^2(\mathbb{R}^d)}$$

implies the bilinear one. The key point is that while the value $q = \frac{d+1}{d-1}$ is the best possible for the linear L^2 estimate, for the bilinear estimate we can obtain some $q < \frac{d+1}{d-1}$. Therefore, interpolating with the trivial estimate

$$\|e^{it\sqrt{-\Delta}} f_1 \; e^{it\sqrt{-\Delta}} f_2\|_{L^\infty(\mathbb{R}^{d+1})} \leq \, C \, \|\widehat{f}_1\|_{L^1(\mathbb{R}^d)} \|\widehat{f}_2\|_{L^1(\mathbb{R}^d)},$$

we get

$$\|e^{it\sqrt{-\Delta}} f_1 \; e^{it\sqrt{-\Delta}} f_2\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq \, C \, \|\widehat{f}_1\|_{L^p(\mathbb{R}^d)} \|\widehat{f}_2\|_{L^p(\mathbb{R}^d)} \tag{0.9}$$

for some $p < 2$. It will be thanks to this that we will be able to answer the first question. Obviously, we need a linear estimate in which we do not have any Fourier angular separation, so how can we use this estimate? Tao, Vargas and Vega [76], using some previous ideas from Bourgain [8], found a method to obtain linear estimates from the bilinear inequalities, using a Whitney decomposition.

Indeed, one can decompose

$$\mathcal{A}_0 \times \mathcal{A}_0 = \bigcup_j \bigcup_{m,m' : \tau_m^{j,0} \sim \tau_{m'}^{j,0}} \tau_m^{j,0} \times \tau_{m'}^{j,0},$$

where $\tau_m^{j,0} \sim \tau_{m'}^{j,0}$ if $|w_m - w_{m'}| \sim 2^{-j}$. That is, for every pair $\tau_m^{j,0}$, $\tau_{m'}^{j,0}$, the angular separation is comparable with the width of the sectors $\tau_m^{j,0}, \tau_{m'}^{j,0}$. Therefore, for $\widehat{f}$ supported in $\mathcal{A}_0$,

$$\begin{aligned}
\|e^{it\sqrt{-\Delta}} f\|^2_{L^{2q}(\mathbb{R}^{d+1})} &= \|e^{it\sqrt{-\Delta}} f \; e^{it\sqrt{-\Delta}} f\|_{L^q(\mathbb{R}^{d+1})} \\
&= \| \sum_j \sum_{m,m' : \tau_m^{j,0} \sim \tau_{m'}^{j,0}} e^{it\sqrt{-\Delta}} P_0 f_m^j \; e^{it\sqrt{-\Delta}} P_0 f_{m'}^j \|_{L^q(\mathbb{R}^d)} \\
&\leq \sum_j \sum_{m,m' : \tau_m^{j,0} \sim \tau_{m'}^{j,0}} \|e^{it\sqrt{-\Delta}} P_0 f_m^j \; e^{it\sqrt{-\Delta}} P_0 f_{m'}^j \|_{L^q(\mathbb{R}^d)},
\end{aligned}$$

where we have used the triangular inequality.

A rescaling argument, which is possible because the angular separation is comparable with the width of the sector, will allow us to use the bilinear estimate. In fact, the argument will be more involved. The triangular inequality will be replaced by a finer estimate that will be explained later.

The following bilinear estimate was proven by Wolff [84].

Theorem 0.2. [84] *Let $\frac{d+3}{d+1} < r_1$, and suppose that $\angle(w_m, w_{m'}) \sim 1$. Then,*

$$\|e^{it\sqrt{-\Delta}}P_0 f^1_m \; e^{it\sqrt{-\Delta}}P_0 f^1_{m'}\|_{L^{r_1}(\mathbb{R}^{d+1})} \leq C \,\|\widehat{P_0 f^1_m}\|_{L^2(\mathbb{R}^d)}\|\widehat{P_0 f^1_{m'}}\|_{L^2(\mathbb{R}^d)}. \tag{0.10}$$

This estimate is sharp except for the endpoint $r_1 = \frac{d+3}{d+1}$ which was proven by Tao [74]. By interpolation, we obtain (0.9) for some $p < 2$.

We have seen how to deduce linear estimates, where there is no angular separation, from bilinear estimates with angular separation. We are assuming that the function $\widehat{f}$ was supported on $\mathcal{A}_0$. For a general function f, we can use Littlewood–Paley theory, that is,

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}f\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} &\leq C\sum_k \|e^{it\sqrt{-\Delta}}P_k f\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\ &= C\sum_k \|e^{it\sqrt{-\Delta}}P_k f \; e^{it\sqrt{-\Delta}}P_k f\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.\end{aligned}$$

By rescaling, we reduce the problem to a situation in which the hypothesis of Theorem 0.2 holds. It turns out that from this we are not able to get $q > 2$ as we want in Theorem 1.1. We need to use the following extension of Theorem 0.2 proved by Tao [74].

Theorem 0.3. [74] *Let $\frac{d+3}{d+1} \leq r_1 \leq 2$, and suppose that $\angle(w_m, w_{m'}) \sim 1$. Then for all $\epsilon > 0$,*

$$\|e^{it\sqrt{-\Delta}}P_0 f^1_m e^{it\sqrt{-\Delta}}P_\ell f^1_{m'}\|_{L^{r_1}(\mathbb{R}^{d+1})} \leq \; C\, 2^{\ell(\frac{1}{r_1}-\frac{1}{2}+\epsilon)}\|\widehat{P_0 f^1_m}\|_{L^2(\mathbb{R}^d)}\|\widehat{P_\ell f^1_{m'}}\|_{L^2(\mathbb{R}^d)}.$$

This can be written in terms of Sobolev norms as

$$\|e^{it\sqrt{-\Delta}}P_0 f^1_m e^{it\sqrt{-\Delta}}P_\ell f^1_{m'}\|_{L^{r_1}(\mathbb{R}^{d+1})} \leq \; C\, 2^{\ell(\frac{1}{r_1}-1+\epsilon)}\|\widehat{P_0 f^1_m}\|_{\dot{H}^{\frac12}(\mathbb{R}^d)}\|\widehat{P_\ell f^1_{m'}}\|_{\dot{H}^{\frac12}(\mathbb{R}^d)}.$$

For ϵ small enough we have that $\frac{1}{r_1} - 1 + \epsilon < 0$, so we get some gain by working in different scales. This idea will allow us to answer the second question. The problem is that now, instead of using Littlewood–Paley, we have

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}f\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} &= \|e^{it\sqrt{-\Delta}}f \; e^{it\sqrt{-\Delta}}f\|^{\frac12}_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\ &= \|\sum_{k>\ell} e^{it\sqrt{-\Delta}}P_k f \; e^{it\sqrt{-\Delta}}P_\ell f + \sum_{k\leq \ell} e^{it\sqrt{-\Delta}}P_k f \; e^{it\sqrt{-\Delta}}P_\ell f\|^{\frac12}_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\ &\leq \; C\,\Big(\sum_{\ell>0}\|\sum_k e^{it\sqrt{-\Delta}}P_k f \; e^{it\sqrt{-\Delta}}P_{k+\ell} f\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\Big)^{\frac12},\end{aligned}$$

and we have to deal with a new summation in ℓ.

The orthogonality property. When we use the Whitney decomposition, we have to deal with

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}P_k f\ e^{it\sqrt{-\Delta}}P_{k+\ell}f\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \\ = \|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k f_m^j\ e^{it\sqrt{-\Delta}}P_{k+\ell}f_{m'}^j\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.\end{aligned}$$

In order to get the refinement, it is not enough to use the triangle inequality. Instead, we prove that the different terms of the summation are essentially disjoint Fourier supported. Indeed, the Fourier transform of $e^{it\sqrt{-\Delta}}P_k f_m^j\ e^{it\sqrt{-\Delta}}P_{k+\ell}f_{m'}^j$ is supported in the set

$$H_{j,m,m'}^{k,\ell} = \left\{(\xi,\tau)\in A_{k+\ell}\times\mathbb{R}:\quad d\big((\xi,\tau),\mathcal{C}\big)\sim 2^{-2j}2^k,\quad \angle(w_m,\xi)\le C\,2^{-j}\right\},$$

with $A_{k+\ell}=\mathcal{A}_{k+\ell-1}\cup\mathcal{A}_{k+\ell}\cup\mathcal{A}_{k+\ell+1}$.

In dimension $d=3$ as $\frac{d+1}{d-1}=2$, by Plancherel identity, this disjointness is enough to prove

$$\begin{aligned}\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k f_m^j\ e^{it\sqrt{-\Delta}}P_{k+\ell}f_{m'}^j\|^2_{L^2(\mathbb{R}^{3+1})} \\ \le C\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\Delta}}P_k f_m^j\ e^{it\sqrt{-\Delta}}P_{k+\ell}f_{m'}^j\|^2_{L^2(\mathbb{R}^{3+1})}.\end{aligned}$$

For other dimensions $d\neq 3$ we need to prove a substitute of the L^2 orthogonality. The first attempt would be to use a lemma due to Tao, Vargas and Vega [76], which give the orthogonality in L^p under the hypothesis of the supports of the functions being contained in a collection of dilates of almost disjoint rectangles. The problem that we encounter is that we cannot find any collection of almost disjoints rectangles $R_{j,m,m'}$ such that $H_{j,m,m'}^{k,\ell}\subset R_{j,m,m'}$.

We need to generalize the lemma to sets different than rectangles. This gives (see Lemma 1.2) an extra factor depending on the annular distance 2^ℓ;

$$\begin{aligned}\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k f_m^j\ e^{it\sqrt{-\Delta}}P_{k+\ell}f_{m'}^j\|^r_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \\ \le C\,2^{c\ell}\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\Delta}}P_k f_m^j\ e^{it\sqrt{-\Delta}}P_{k+\ell}f_{m'}^j\|^r_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.\end{aligned}$$

for some $c>0$ and $r>1$.

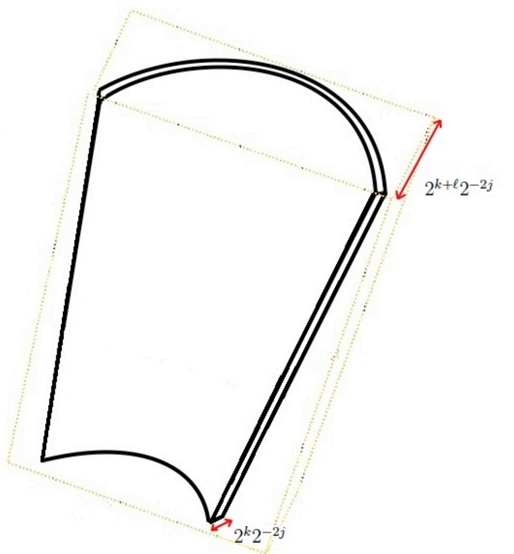

FIGURE 0.1. Due to the curvature of the cone the side of the minimal rectangle $R_{j,m,m'}$ containing $H^{k,\ell}_{j,m,m'}$ is 2^ℓ times the width of $H^{k,\ell}_{j,m,m'}$.

Summing the pieces. We have broken the frequency space in annuli and different angular scales interacting in a bilinear way. Eventually we need to recover the full frequency space. One of the main problems is that we have incurred a factor of $2^{c\ell}$ which complicates the summation. The idea is the following: taking α very small, we divide the terms

$$\begin{aligned}
&\|\sum_j \sum_{m,m':\tau^{j,k}_m\sim\tau^{j,k+\ell}_{m'}} e^{it\sqrt{-\Delta}}P_k f^j_m\, e^{it\sqrt{-\Delta}}P_{k+\ell} f^j_{m'}\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\
&\qquad\le \|\sum_j \sum_{m,m':\tau^{j,k}_m\sim\tau^{j,k+\ell}_{m'}} e^{it\sqrt{-\Delta}}P_k f^j_m\, e^{it\sqrt{-\Delta}}P_{k+\ell} f^j_{m'}\|^{\alpha}_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\
&\qquad\times \|\sum_j \sum_{m,m':\tau^{j,k}_m\sim\tau^{j,k+\ell}_{m'}} e^{it\sqrt{-\Delta}}P_k f^j_m\, e^{it\sqrt{-\Delta}}P_{k+\ell} f^j_{m'}\|^{1-\alpha}_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.
\end{aligned}$$

For the first term we use the orthogonality that we explained in the previous subsection. With α sufficiently small we will be able sum.

For the second term we use orthogonality at each single scale, that is for each fixed j. This does not give any extra factor of $2^{c\ell}$, but on the other hand we cannot get the refinement from this term. Nevertheless we can still get the L^2 norm, thanks to an atomic decomposition proven by Keel and Tao [39].

0.2. The linear profile decomposition

What is a linear profile decomposition?

We recall the definition of a compact operator.

Definition 0.1. *Let $F : X \to Y$ be a linear operator between two Banach spaces. F is compact if for every bounded sequence $x_n \in X$, the sequence $F(x_n)$ has a convergent subsequence.*

Consider the wave operator,

$$\begin{aligned} S : \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} &\longrightarrow \; L^{2\frac{d+1}{d-1}} \\ (u_0, u_1) \quad &\longrightarrow \; S[u_0, u_1]. \end{aligned}$$

It is not difficult to see that this is not a compact operator. Indeed, letting $u_0, u_1 \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, we define $u_{0,n}(x) = n^{\frac{d-1}{2}} u_0(nx)$, $u_{1,n}(x) = n^{\frac{d+1}{2}} u_1(nx)$, $n \in \mathbb{N}$. We have that

$$\|u_{0,n}\|_{\dot{H}^{\frac{1}{2}}} = \|u_0\|_{\dot{H}^{\frac{1}{2}}}, \quad \|u_{1,n}\|_{\dot{H}^{-\frac{1}{2}}} = \|u_1\|_{\dot{H}^{-\frac{1}{2}}},$$

and, in particular, the sequence is bounded in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$. However

$$S[u_{0,n}, u_{1,n}](x,t) = n^{\frac{d-1}{2}} S[u_0, u_1](nx, nt)$$

does not have any convergent subsequence.

It is natural, therefore, to consider for which kind of sequences the wave operator loses its compactness. It turns out, that the defect of compactness comes always from the symmetries of the wave equation. Letting $(u_0, u_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, the following transformations of $S[u_0, u_0](x,t)$ are also solutions of the wave equation

Dilation: $u(x,t) = r_1 \, S[u_0, u_1](r_2 x, r_2 t)$ with $r_1, r_2 > 0$.

Space-time translation: $u(x,t) = S[u_0, u_1](x + x_0, t + t_0)$ with $x_0 \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

Lorentz transformation: $u(x,t) = S[u_0, u_1](x - x_v + \frac{x_v - vt}{\sqrt{1-|v|^2}}, \frac{t - vx}{\sqrt{1-|v|^2}})$ with $v \in \mathbb{R}^d$, $|v| < 1$ and $x_v \in \mathbb{R}^d$ is the projection of x onto the line parallel to v.

Rotation: $u(x,t) = S[u_0, u_1](\theta x, t)$ with $\theta \in SO(d)$.

Phase shifts: $u(x,t) = e^{\theta_+ i} S_+[u_0, u_1](x,t) + e^{\theta_- i} S_-[u_0, u_1](x,t)$ with $\theta_+, \theta_- \in [0, 2\pi)$, where

$$S_\pm[u_0, u_1](t) = \frac{1}{2}\big(e^{\pm it\sqrt{-\Delta}} u_0 \pm \frac{1}{i} \frac{e^{\pm it\sqrt{-\Delta}} u_1}{\sqrt{-\Delta}}\big).$$

The last two types of symmetry do not violate the compactness of the wave operator (indeed, they are a compact group of transformations), while the combination of the first three can do so.

More precisely, letting $(r^n, \ell^n, w^n, x^n, t^n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, we define the transformations Γ^n by

$$\Gamma^n f(x,t) = \left(\frac{r^n}{\ell^n}\right)^{\frac{d-1}{2}} f\Big((T^{\ell^n}_{w^n})^{-1} r^n (x - x^n, t - t^n)\Big),$$

where $(T^{\ell^n}_{w^n})^{-1}$ is a transformation defined by scaling by ℓ^n after a Lorentz transformation with $v = (w^n, 1)$. We write also

$$S[\Gamma^n_0 u_{0,n}, \Gamma^n_1 u_{1,n}] := \Gamma^n S[u_{0,n}, u_{1,n}].$$

The defect of compactness comes always from these transformations. To see this we recall the following characterization of compactness:

Let X be reflexive, $F : X \to Y$ is a compact operator if and only if $x_n \not\rightharpoonup 0$ whenever $\|F(x_n)\| \not\to 0$.

Through this characterization we can state precisely that the transformations Γ^n are the only responsible for the defect of compactness for the wave operator.

Proposition 2.1. *Let $d \geq 2$, and let $(u_{0,n}, u_{1,n})_n$ be a sequence in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ such that*

$$\|(u_{0,n}, u_{1,n})\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} \leq M \quad \textit{and} \quad \|S[u_{0,n}, u_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq A.$$

Then, there exists a sequence $(r^n, \ell^n, w^n, x^n, t^n)$ in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$ such that, up to a subsequence,[2]

$$((\Gamma^n_0)^{-1} u_{0,n}, (\Gamma^n_1)^{-1} u_{1,n}) \underset{n\to\infty}{\rightharpoonup} (\mathtt{U_0}, \mathtt{U_1}) \quad \textit{with} \quad \|(\mathtt{U_0}, \mathtt{U_1})\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} \geq C(A, M).$$

As a consequence of this result, we can also prove a profile decomposition for the wave equation. Roughly speaking, the profile decomposition states that for any bounded sequence $(u_{0,n}, u_{1,n}) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, we have that $\{S[u_{0,n}, u_{1,n}]\}_n$ can be written, up to taking a subsequence, as a sum of our transformations $\{\{\Gamma^n_j S[\phi^j_0, \phi^j_1]\}_n\}_j$, called profiles which do not interact much, and a remainder term r^N_n which is small in some sense:

$$S[u_{0,n}, u_{1,n}] = \sum_{j=1}^{N} \Gamma^n_j S[\phi^j_0, \phi^j_1] + r^N_n.$$

[2]Proposition 2.1 in Chapter 2 is formulated in a different way, but it is equivalent to the one given here (see Lemma 2.3).

The precise theorem proved in the second chapter is stated as follows.

Theorem 2.1. *Let* $(u_{0,n}, u_{1,n})_n$ *be a bounded sequence in* $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ *with* $d \geq 2$. *Then there exist a subsequence (still denoted by* $(u_{0,n}, u_{1,n})_n$*), a sequence* $(\phi_0^j, \phi_1^j)_{j\in\mathbb{N}} \subset \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ *and a family of orthogonal sequences* $\{(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{n\in\mathbb{N}}\}_{j\in\mathbb{N}}$ *in* $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, *such that for every* $N \geq 1$,

$$S[u_{0,n}, u_{1,n}](x,t) = \sum_{j=1}^{N} \Gamma_j^n S[\phi_0^j, \phi_1^j](x,t) + S[R_{0,n}^N, R_{1,n}^N](x,t), \tag{0.11}$$

with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S[R_{0,n}^N, R_{1,n}^N]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = 0. \tag{0.12}$$

Furthermore, for every $N \geq 1$, *as* $n \to \infty$

$$\|(u_{0,n}, u_{1,n})\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} = \sum_{j=1}^{N} \|(\phi_0^j, \phi_1^j)\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + \|(R_{0,n}^N, R_{1,n}^N)\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + o(1).$$

The definition of orthogonal sequences is given at the begining of the second chapter.

The first profile decomposition for the wave equation is due to Bahouri and Gérard [1], with data in $\dot{H}^1 \times L^2$, and to Merle and Vega [58] for the L^2 Schrödinger equation. The first one uses a Sobolev inequality and is based on the ideas in [33]; and the second one uses a Strichartz refinement and is based on the ideas in [11].

Using the Sobolev method, Keraani [45] proved a profile decomposition for the $\dot{H}^1$ Schrödinger equation in dimension $d \geq 3$. Bulut [13] obtained a similar result for the $\dot{H}^1 \times L^2$ wave equation in dimensions $d \geq 3$, and Fanelli and Visciglia [29] for a large clase of a dispersive propagators.

Using the Strichartz method, which requires a Strichartz refinement, Carles and Keraani [14] proved the profile decomposition for the L^2 Schrödinger equation in dimension $d = 1$, and Bégout and Vargas [3] in dimension $d \geq 3$. Shao [69] obtained the result for the Airy equation, and Killip, Stovall and Visan [48] for the $\dot{H}^1 \times L^2$ Klein–Gordon equation.

Our argument belongs to the second group and the Strichartz refinement is the basic ingredient.

As an application of this profile decomposition we can prove that there exist maximizers for the inequality (0.7).

Corollary 2.1. *Let* $d \geq 2$. *Then there exists a maximizing pair* $(\psi_0, \psi_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ *such that*

$$\|S[\psi_0, \psi_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = W(d)\|(\psi_0, \psi_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)},$$

where

$$W(d) := \sup \Big\{ \|S[\phi_0,\phi_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} : (\phi_0,\phi_1) \in \dot H^{\frac12} \times \dot H^{-\frac12} \text{ with } \|(\phi_0,\phi_1)\|_{\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)} = 1 \Big\}.$$

A lot of related work has been carried out in recent years. Kunze [56] proved the existence of maximizers for the Strichartz inequality in the case of the Schrödinger equation with $d = 1$. Foschi [31] calculated the maximizers for the Schrödinger equation in dimensions $d = 1, 2$. Hundertmark and Zharnitsky [37], Bennett, Bez, Carbery and Hundertmark [4], and Carneiro [15] reproved this result. Foschi also calculated the maximizers for the wave equation with $d = 3$. Bez and Rogers [6] found the maximizers for the energy-Strichartz inequality for the $\dot H^1 \times L^2$ wave equation with $d = 5$. Shao [69] proved that maximizers exist for the case of the Schrödinger equation in all dimensions, and Bulut [13] for the $\dot H^s \times \dot H^{s-1}$ wave equation with $s \geq 1$ with $d \geq 3$, both with the profile decomposition. Christ and Shao [18, 19] found the maximizers for the dual restriction inequality for the sphere with $d = 2$, and Fanelli, Vega and Visciglia [27] extended this to nonendpoint inequalities for more general surfaces and dimensions. Finally, Quilodrán [62] proved the nonexistence of maximizers for the dual restriction inequality for the hyperboloid.

Outline of the proof

Informally the proof goes through the following steps:

Step 1: Obtaining a profile decomposition assuming that the initial data is bounded and with compact Fourier support away from zero. The transformations on the profiles are not the transformations Γ^n, but just space-time translations. The key point in this step is an improvement of the restriction theorem of (0.5), proved by Wolff [84] (the linear version of Theorem 0.2).

Step 2: Reduce to the case when the initial data is bounded and has compact Fourier support away from zero, but with an epsilon dependence. That is, for a fixed $\epsilon > 0$ we find some profiles with a remainder term smaller than ϵ in the Strichartz norm and constants depending on ϵ.

The reduction to the case when the initial data is bounded and has compact Fourier support is the main step and is based in the ideas of Bourgain in [11]. We use here our Strichartz refinement Theorem 1.1. The idea is that if $\epsilon < \|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}}$, then the Strichartz refinement ensures that there should be a dominant sector $\tau_m^{j,k}$ where the Fourier support of the initial data is also big enough[3]:

$$\|\widehat{P_k(u_0)}^j_m\|_p + \|\widehat{P_k(u_1)}^j_m\|_p \geq C(\epsilon, \|(u_0,u_1)\|_{\dot H^{\frac12}\times \dot H^{-\frac12}}) |\tau_m^{j,k}|^\theta .$$

[3] Roughly speaking, the Besov improvement permits to get functions with compact Fourier support, and the L^p with $p < 2$ improvement permits to have the functions bounded in L^∞.

for some θ. This permits, after some calculations to find a family of functions $\left\{(f^i_{0,n}, f^i_{1,n})\right\}_{1\le i\le N}$ bounded and compact Fourier supported away from the origin such that

$$\|S[u_{0,n}, u_{1,n}] - \sum_{i}^{N} S[f^i_{0,n}, f^i_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \epsilon,$$

where N depends only on ϵ and $\|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}}$. This epsilon dependence on the family, is the reason for which we need one more step.

Another key point here, which is based in Merle and Vega [58], is to establish the following dichotomy for every pair $1 \le i, j \le N$ with $i \ne j$:

- Either the sequence of functions $(f^i_{0,n}, f^i_{1,n})$ and $(f^j_{0,n}, f^j_{1,n})$ can be added preserving the properties of compact Fourier support and boundedness.

- Or both sequences are orthogonal in some sense as $n \to \infty$.

This step yields Proposition 2.1.

Step 3: Remove the ϵ dependance using Proposition 2.1.

0.3. Norm concentration for the nonlinear wave equation

Consider the nonlinear wave equation in $\mathbb{R}^{d+1}$ with $d \ge 3$:

$$\begin{cases} u_{tt} - \Delta u + \gamma u|u|^p = 0 \\ u(0) = u_0 \in \dot{H}^s,\ u_t(0) = u_1 \in \dot{H}^{s-1}, \end{cases} \tag{0.13}$$

with $\gamma \in \mathbb{R}\setminus\{0\}$.

The equation is invariant under the scaling

$$u_r(x,t) = r^{\frac{2}{p}} u(rx, rt).$$

This invariance determines the critical Sobolev space for the initial data (u_0, u_1). We want

$$\|u_r(0)\|_{\dot{H}^s} = \|u(0)\|_{\dot{H}^s}, \quad \|\partial_t u_r(0)\|_{\dot{H}^{s-1}} = \|\partial_t u(0)\|_{\dot{H}^{s-1}}.$$

A calculation shows that the critical regularity corresponds to the case where

$$s_c = \frac{d}{2} - \frac{2}{p}.$$

Therefore our problem is critical if the initial data is in $(\dot{H}^{s_c} \times \dot{H}^{s_c-1})$.

The energy $E(u)$ is conserved, where

$$E(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \gamma\frac{1}{2}|u|^{p+2}dx.$$

We say that the equation is defocusing if $\gamma < 0$ and focusing if $\gamma > 0$.

In the last chapter, we consider the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ critical equation

$$\begin{cases} u_{tt} - \Delta u + \gamma u|u|^{\frac{4}{d-1}} = 0 \\ u(0) = u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d),\ u_t(0) = u_1 \in \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d), \end{cases} \tag{0.14}$$

for $\gamma \in \mathbb{R} \setminus \{0\}$ and $d \geq 3$.

We notice that the we cannot use the energy since our solution is not regular enough.

What is norm concentration?

It is not known if this equation is globally well-posed[4] for general initial data. One possible strategy to try to prove such result would be to show that if the solution blows up in finite time, then it concentrates in some sense. Then, if one could prove that this concentration cannot take place, then the solution cannot blow up. Hence, the equation should be globally well-posed.

Our contribution goes in this direction. We prove that if the solution blows up, then the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ norm of the solution concentrates in space time as the solution approaches the blow up time. This is achieved under the assumption that the norm of the solution is bounded in the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ norm. This assumption, the so called *type II blow-up*, has been considered recently by many authors: Krieger, Schlag and Tataru [55] constructed type II blow-up solutions for the focusing energy critical wave equation in dimension $d = 3$. Kenig and Merle [43], and Killip and Visan [53], [54], proved for the energy supercritical wave equation that type II blow-up cannot happen, using a contradiction in the spirit of the one explained before. Duyckaerts, Kenig and Merle [24], [25] characterized some properties of solutions with type II blow up.

We denote by $R^{j,k}$ a rectangle in $\mathbb{R}^d$ of dimensions $2^{-k} \times 2^{-k}2^j \times \cdots \times 2^{-k}2^j$, with $k \in \mathbb{R}$, $j \in \mathbb{R}^+$.

The main result in Chapter 3 is the following

Theorem 3.1. *Suppose that u is a solution of (0.14) that blows up at $T_{max} < \infty$. Suppose also that*

$$\sup_{t\in[0,T_{\max})} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t u(t)\|_{\dot{H}^{-\frac{1}{2}}} \leq B. \tag{0.15}$$

[4]The precise notion of well-posed and blow-up solutions will be settled in Chapter 3.

Then[5]

$$\limsup_{t\to T_{\max}} \sup_{R^{j,k}\colon T_{\max}-t\geq 2^{-k}2^{2j}} \|\chi_{R^{j,k}}u(t)\|_{\dot H^{\frac12}} + \|\chi_{R^{j,k}}\partial_t u(t)\|_{\dot H^{-\frac12}} > \epsilon, \tag{0.16}$$

where ϵ *depends only on* B *and* γ*.*

The proof relies on an argument of Bourgain [11] for the Schrödinger equation. The difficulties here come from the fact that we have to concentrate in space-time at the same time as controlling the frequency. This is caused by the fact that we work in Sobolev spaces rather than in L^2.

The main tool in this chapter is the Strichartz refinement Theorem 1.1.

The linear profile decomposition will be explored in a forthcoming work [66] to prove a nonlinear profile decomposition for (0.14), which allows us to prove that if there are blow-up solutions, then there is a blow-up solution with minimal initial data. This follows the ideas of Keraani [45] and again could be useful in ruling out the possibility of blow-up, as it was for the Schrödinger equation.

The results of the first two chapters are published in [64] and the result of the last chapter has been submitted for publication [65].

[5]Theorem 3.1 in Chapter 3 is formulated in a stronger way.

CHAPTER 1

The Strichartz refinement

1.1. Introduction

The wave equation $\partial_{tt}u = \Delta u$, in $\mathbb{R}^{d+1}$, with initial data $u(\cdot, 0) = u_0$, $\partial_t u(\cdot, 0) = 0$, has solution which can be written as

$$u(\cdot, t) = \frac{1}{2}\big(e^{it\sqrt{-\Delta}}u_0 + e^{-it\sqrt{-\Delta}}u_0\big),$$

where

$$e^{\pm it\sqrt{-\Delta}}u_0(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{u_0}(\xi)e^{i(x\cdot\xi \pm t|\xi|)}d\xi.$$

In 1977, Strichartz [73] proved (see also [81]), his fundamental inequality

$$\|e^{it\sqrt{-\Delta}}g\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}, \tag{1.1}$$

where $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ denotes the homogeneous Sobolev space with half a derivative in $L^2(\mathbb{R}^d)$.

A consequence of our work will be the Besov space refinement

$$\|e^{it\sqrt{-\Delta}}g\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\|g\|_{\dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^d)}, \tag{1.2}$$

where $q = 2\frac{d+1}{d-1}$ for $d \geq 3$, and $q = 3$ for $d = 2$. Here $\dot{B}^s_{2,q}$ is defined by

$$\|g\|_{\dot{B}^s_{2,q}} = (\sum_k 2^{ksq}\|P_k g\|_2^q)^{\frac{1}{q}},$$

where $\widehat{P_k f} = \chi_{\mathcal{A}_k}\widehat{f}$ and $\mathcal{A}_k = \{\xi \in \mathbb{R}^d;\ \ 2^k \leq |\xi| \leq 2^{k+1}\}$. The Strichartz estimate (1.1) follows from (1.2) by the sequence space embedding $\ell^2 \hookrightarrow \ell^q$.

For our applications the following refinement will be of more use. Let $M = \{w_m\}_m \subset \mathbb{S}^{d-1}$ be maximally 2^{-j}-separated, and define $\tau^{j,k}_m$ by

$$\tau^{j,k}_m := \left\{\xi \in \mathcal{A}_k : \big|\frac{\xi}{|\xi|} - w_m\big| \leq \big|\frac{\xi}{|\xi|} - w_{m'}\big| \ \text{ for every } w_{m'} \in M,\ m' \neq m\right\}.$$

Note that $\big|\frac{\xi}{|\xi|} - w_m\big| \leq 2^{-j}$ for all $\xi \in \tau^{j,k}_m$. We also set $\widehat{P_k g^j_m} = \chi_{\tau^{j,k}_m}\widehat{g}$.

Theorem 1.1. *Let $d \geq 2$, $q = 2\frac{d+1}{d-1}$. Then, there exist some $p < 2$ and some $\theta > 0$ such that*[1]

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \sup_{j,k,m} 2^{k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^\theta \|g\|_{\dot{H}^{\frac{1}{2}}}^{1-\theta}. \tag{1.3}$$

This kind of refinement was obtained for the Schrödinger equation by Moyua, Vargas and Vega [59], [60] for dimension $d = 2$, generalizing and improving a result of Bourgain [8], by Carles and Keraani [14] for dimension $d = 1$, and by Bégout and Vargas [3] for dimensions $d \geq 3$. See also Rogers and Vargas [67] for the nonelliptic Schrödinger equation, and Chae, Hong and Lee [16] for higher order Schrödinger equations.

It is a relatively simple task to adapt the arguments of [3] in order to prove Theorem 1.1 for functions which are Fourier supported in dyadic annuli (see [34]). These estimates can be combined, via the Littlewood–Paley inequality, to obtain

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Big(\sum_k 2^k \big(\sum_j \sum_m |\tau_m^{j,k}|^{q\frac{p-2}{2p}} \|\widehat{P_k g_m^j}\|_p^q\big)^{\frac{2}{q}}\Big)^{\frac{1}{2}}. \tag{1.4}$$

This does not yield the refinement (1.3), and perhaps more importantly, it does not yield the profile decomposition which is the principal result of the second chapter because it is not possible to take a supremum in k without losing some regularity. In order to prove Theorem 1.1, we deal with the interaction between dyadic annuli by combining Tao's bilinear inequality [74] (which improved upon Wolff's estimate [84]) with what is perhaps a new orthogonality property for the cone.

When $d \neq 3$, we are lead to consider orthogonality properties of thickened pieces of the cone in L^p, which is a deep and largely unanswered question (see for example [83] or [32]). We sidestep the problem by strengthening the standard lemma which proves that the norm on the right hand side of (1.4) is smaller than the $\dot{H}^{1/2}$ norm. This is achieved using an atomic decomposition of L^p due to Keel and Tao [39].

In chapter 2 and 3 we will be using the Strichartz refinement.

After the results of this chapter were submitted for publication in [64], Quilodrán posted a similar result to Theorem 1.1 for the case of dimension $d = 2$ on the arXiv [63].

[1] The expression $A \lesssim B$ denotes $A \leq CB$, where the value of the positive constant C will change from line to line. The expression $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

1.2. The Strichartz refinement

Theorem 1.1 and the estimate (1.2) easily follow from the following theorems. We define the $X^k_{p,q}$-norm by

$$\|f\|_{X^k_{p,q}} = (\sum_j \sum_m |\tau^{j,k}_m|^{q\frac{p-2}{2p}} \|\widehat{P_k f^j_m}\|^q_p)^{\frac{1}{q}}.$$

The case $d = 3$ will be easier thanks to some extra orthogonality.

Theorem 1.2. *Let $\frac{8}{5} < p < 2$. Then, for all $0 \leq \theta < \frac{1}{2}$,*

$$\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} \lesssim (\sum_k 2^{2k}\|P_k g\|^4_{X^k_{p,4}})^{\frac{1}{4}} \lesssim \sup_{j,k,m} 2^{k\frac{\theta}{2}}|\tau^{j,k}_m|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k g^j_m}\|^\theta_p \|g\|^{1-\theta}_{\dot{B}^{\frac{1}{2}}_{2,4(1-\theta)}}.$$

For dimensions $d = 2$ and $d \geq 4$, we will prove

Theorem 1.3. *Let $\frac{5}{3} < \lambda < 2$ and set $p = \frac{6}{6-\lambda}$. Then, for all $0 \leq \theta < \lambda - \frac{5}{3}$,*

$$\|e^{it\sqrt{-\Delta}}g\|_{L^6(\mathbb{R}^{2+1})} \lesssim \sup_{j,k,m} 2^{k\frac{\theta}{2}}|\tau^{j,k}_m|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k g^j_m}\|^\theta_p \|g\|^{1-\theta}_{\dot{B}^{\frac{1}{2}}_{2,3(1-\theta)}}.$$

Let $d > 3$, $q = \frac{2(d+1)}{d-1}$ and $\frac{d+3}{d+1} < \lambda < \frac{d+1}{d-1}$. Set $p = \frac{2(d+1)}{2(d+1)-\lambda(d-1)}$. Then, for all $0 \leq \theta < \lambda\frac{2}{d-3} - \frac{2(d+3)}{(d-3)(d+1)}$,

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \sup_{j,k,m} 2^{k\frac{\theta}{2}}|\tau^{j,k}_m|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k g^j_m}\|^\theta_p \|g\|^{1-\theta}_{\dot{B}^{\frac{1}{2}}_{2,q(1-\theta)}}.$$

The main tool will be Tao's bilinear estimate, proved in [74], which improved upon Wolff's theorem in [84] (see also [9]).

Theorem 1.4. [74] *Let $\frac{d+3}{d+1} \leq r_1 \leq 2$, and suppose that $\angle(w_m, w_{m'}) \sim 1$. Then for all $\epsilon > 0$,*

$$\|e^{it\sqrt{-\Delta}}P_0 g^1_m e^{it\sqrt{-\Delta}}P_\ell g^1_{m'}\|_{L^{r_1}(\mathbb{R}^{d+1})} \lesssim 2^{\ell(\frac{1}{r_1}-\frac{1}{2}+\epsilon)}\|\widehat{P_0 g^1_m}\|_{L^2(\mathbb{R}^d)}\|\widehat{P_\ell g^1_{m'}}\|_{L^2(\mathbb{R}^d)}. \tag{1.5}$$

By a rescaling argument (see [76] and [84]) and interpolation we get the following corollary. We include the proof for the benefit of the reader.

Corollary 1.1. *Let $\frac{d+3}{d+1} \le r_1 \le 2$, $r \ge r_1$, and suppose that $\angle(w_m, w_{m'}) \sim 2^{-j}$. Then for all $\epsilon > 0$,*

$$\begin{aligned}&\|e^{it\sqrt{-\Delta}} P_k g_m^j e^{it\sqrt{-\Delta}} P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}\\ &\qquad \lesssim 2^{\ell(\frac{1}{r}-\frac{r_1}{2r}+\epsilon)} 2^{k(\frac{r_1 d}{r}-\frac{d+1}{r})} 2^{j(\frac{d+1}{r}-\frac{r_1(d-1)}{r})} \|\widehat{P_k g_m^j}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)} \|\widehat{P_{k+\ell} g_{m'}^j}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)}.\end{aligned}$$

The following remark will be useful for the proof.

Remark 1.1. *Setting $\widehat{g}(\xi) = f(\xi, |\xi|)$ and $d\sigma(\xi,\tau) = \delta(|\xi| - \tau) d\xi$, we have that*

$$\begin{aligned} e^{it\sqrt{-\Delta}} g(x) &= \frac{1}{(2\pi)^d} \int e^{i(x\cdot\xi + t|\xi|)} \widehat{g}(\xi) d\xi \\ &= \frac{1}{(2\pi)^d} \int e^{i(x\cdot\xi + t|\xi|)} f(\xi, |\xi|) d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{C}} e^{i(x\cdot\xi + t\tau)} f(\xi,\tau) d\xi d\tau \\ &= \widehat{f d\sigma}(x,t), \end{aligned}$$

where $\mathbf{C} := \left\{(\xi,\tau) \in \mathbb{R}^{d+1} : \quad |\xi| = \tau\right\}$.

Therefore, if $\widehat{g}$ is supported in $\tau_m^{j,k}$, we can interpret $e^{it\sqrt{-\Delta}} g$ as the Fourier transform of a measure supported in

$$\tilde{\tau}_m^{j,k} := \left\{(\xi, |\xi|) \in \mathbb{R}^{d+1} : \quad \xi \in \tau_m^{j,k}\right\}.$$

Proof of Corollary 1.1. We have the trivial estimate

$$\|e^{it\sqrt{-\Delta}} P_0 f_n^1 e^{it\sqrt{-\Delta}} P_\ell f_{n'}^1\|_{L^\infty(\mathbb{R}^{d+1})} \lesssim \|\widehat{P_0 f_n^1}\|_{L^1(\mathbb{R}^d)} \|\widehat{P_\ell f_{n'}^1}\|_{L^1(\mathbb{R}^d)}.$$

By interpolation with (1.5) we get for $r \ge r_1 \ge \frac{d+3}{d+1}$,

$$\|e^{it\sqrt{-\Delta}} P_0 f_n^1 e^{it\sqrt{-\Delta}} P_\ell f_{n'}^1\|_{L^r(\mathbb{R}^{d+1})} \lesssim 2^{\ell(\frac{1}{r}-\frac{r_1}{2r}+\epsilon)} \|\widehat{P_0 f_n^1}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)} \|\widehat{P_\ell f_{n'}^1}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)}, \tag{1.6}$$

for every $f_n^1, f_{n'}^1$ with $\angle(w_n, w_{n'}) \sim 1$. Letting $w \in \mathbb{S}^{d-1}$, and $j \in [0,\infty)$, we define the transformations $T_w^{2^j}$, which are the composition of a dilation and a Lorentz transformation[2], to be the linear map which preserves the cone and satisfies

$$\begin{aligned} &T_w^{2^j}(w,1) = (w,1), \\ &T_w^{2^j}(w,-1) = 2^{2j}(w,-1), \\ &T_w^{2^j}(x,t) = 2^j(x,t) \ \text{ if } (x,t) \in \mathbb{R}^{d+1} \text{ is orthogonal to } (w,1) \text{ and } (w,-1). \end{aligned} \tag{1.7}$$

[2]The Lorentz transformation $L_w^{2^j}$ is defined by $L_w^{2^j}(x,t) = T_w^{2^j}(2^{-j}(x,t))$.

We have that

$$\det T_w^{2^j} = 2^{j(d+1)}$$

and that if $\tilde{\tau}_m^{j,k}$, $\tilde{\tau}_{m'}^{j,k+\ell}$ with $\angle(w_m, w_{m'}) \sim 2^{-j}$, then, there are $n = n(j,m)$, $n' = n'(j,m,m')$ with $\angle(w_n, w_{n'}) \sim 1$ such that $T_{w_m}^{2^j}(\tilde{\tau}_m^{j,k}) \subset C\tilde{\tau}_n^{1,k}$, $T_{w_m}^{2^j}(\tilde{\tau}_{m'}^{j,k+\ell}) \subset C\tilde{\tau}_{n'}^{1,k+\ell}$ with C a constant depending on the dimension d. By a change of variables

$$\begin{aligned}
e^{it\sqrt{-\Delta}}P_k g_m^j(x) &= \frac{1}{(2\pi)^d}\int e^{i(x\cdot\xi+t|\xi|)}\chi_{\tau_m^{j,k}}(\xi)\widehat{g}(\xi)d\xi \\
&= \frac{2^{kd}C^d}{(2\pi)^d}\int e^{i(2^kCx\cdot\xi+2^kCt|\xi|)}\chi_{C^{-1}\tau_m^{j,0}}(\xi)\widehat{g}(2^kC\xi)d\xi \\
&= \frac{2^{kd}C^d}{(2\pi)^d}\int e^{i\langle((T_{w_m}^{2^j})^{-1}(2^kCx,2^kCt)),(\xi,|\xi|)\rangle}\chi_{\tau_n^{1,0}}(\xi) \\
&\qquad \times |J(T_{0,w_m}^{2^j})^{-1}(\xi)|\chi_{C^{-1}\tau_m^{j,0}}\big((T_{0,w_m}^{2^j})^{-1}\xi\big)\widehat{g}\big((T_{0,w_m}^{2^j})^{-1}2^kC\xi\big)d\xi \\
&= 2^{kd}C^d\, e^{it'\sqrt{-\Delta}}P_0 f_n^1(x'),
\end{aligned}$$

and arguing in the same way

$$e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j(x) = 2^{kd}C^d\, e^{it'\sqrt{-\Delta}}P_\ell f_{n'}^1(x'),$$

where

$$(x',t') = (T_{w_m}^{2^j})^{-1}(2^kCx, 2^kCt),$$

$$\widehat{P_0 f_n^1}(\xi) = \chi_{\tau_n^{1,0}}(\xi)|J(T_{0,w_m}^{2^j})^{-1}(\xi)|\chi_{C^{-1}\tau_m^{j,0}}\big((T_{0,w_m}^{2^j})^{-1}\xi\big)\widehat{g}\big((T_{0,w_m}^{2^j})^{-1}2^kC\xi\big),$$

$$\widehat{P_\ell f_{n'}^1}(\xi) = \chi_{\tau_{n'}^{1,\ell}}(\xi)|J(T_{0,w_m}^{2^j})^{-1}(\xi)|\chi_{C^{-1}\tau_{m'}^{j,\ell}}\big((T_{0,w_m}^{2^j})^{-1}\xi\big)\widehat{g}\big((T_{0,w_m}^{2^j})^{-1}2^kC\xi\big),$$

$$\angle(w_n, w_{n'}) \sim 1,$$

$(T_{0,w_m}^{2^j})^{-1}$ is the transformation defined as

$$(T_{w_m}^{2^j})^{-1}(\xi,|\xi|) = \big((T_{0,w_m}^{2^j})^{-1}(\xi), \big|(T_{0,w_m}^{2^j})^{-1}(\xi)\big|\big),$$

and $|J(T_{0,w_m}^{2^j})^{-1}(\xi)|$ is the jacobian of the transformation $(T_{0,w_m}^{2^j})^{-1}$.

It is easy to see that $|J(T_{0,w_m}^{2^j})^{-1}(\xi)| \sim 2^{-j(d-1)}$ for $\xi \in \tau_n^{j,0}, \tau_{n'}^{j,\ell}$. Therefore we have

$$\|\widehat{P_0 f_n^1}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)} \sim 2^{j(d-1)(\frac{2r-r_1}{2r}-1)}2^{-kd\frac{2r-r_1}{2r}}\|\widehat{P_k g_m^j}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)},$$

$$\|\widehat{P_\ell f_{n'}^1}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)} \sim 2^{j(d-1)(\frac{2r-r_1}{2r}-1)}2^{-kd\frac{2r-r_1}{2r}}\|\widehat{P_{k+\ell}g_{m'}^j}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)},$$

and

$$\begin{aligned}
&\|e^{it\sqrt{-\Delta}}P_k g_m^j\, e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})} \\
&\qquad \sim 2^{k(2d-\frac{(d+1)}{r})}2^{j\frac{(d+1)}{r}}\|e^{it\sqrt{-\Delta}}P_0 f_n^1 e^{it\sqrt{-\Delta}}P_\ell f_{n'}^1\|_{L^r(\mathbb{R}^{d+1})}.
\end{aligned}$$

By (1.6) we get then

$$\begin{aligned}
&\|e^{it\sqrt{-\Delta}}P_k g^j_m \, e^{it\sqrt{-\Delta}}P_{k+\ell}g^j_{m'}\|_{L^r(\mathbb{R}^{d+1})} \\
&\qquad \sim 2^{k(2d-\frac{(d+1)}{r})}2^{j\frac{(d+1)}{r}}\|e^{it\sqrt{-\Delta}}P_0 f^1_n e^{it\sqrt{-\Delta}}P_\ell f^1_{n'}\|_{L^r(\mathbb{R}^{d+1})} \\
&\qquad \lesssim 2^{2kd}2^{j\frac{(d+1)}{r}}2^{\ell(\frac{1}{r}-\frac{r_1}{2r}+\epsilon)}\|\widehat{P_0 f^1_n}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)}\|\widehat{P_\ell f^1_{n'}}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)} \\
&\qquad \sim 2^{\ell(\frac{1}{r}-\frac{r_1}{2r}+\epsilon)}2^{k(\frac{r_1 d}{r}-\frac{d+1}{r})}2^{j(\frac{d+1}{r}-\frac{r_1(d-1)}{r})}\|\widehat{P_k g^j_m}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)}\|\widehat{P_{k+\ell} g^j_{m'}}\|_{L^{\frac{2r}{2r-r_1}}(\mathbb{R}^d)},
\end{aligned}$$

and we deduce the result.

□

1.3. Proof of Theorem 1.2

Proof of Theorem 1.2. We begin by decomposing our estimate in annular pieces in the bilinear setting,

$$\begin{aligned}
\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} &= \|e^{it\sqrt{-\Delta}}g \, e^{it\sqrt{-\Delta}}g\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{3+1})} \\
&= \|\sum_{k>\ell} e^{it\sqrt{-\Delta}}P_k g \, e^{it\sqrt{-\Delta}}P_\ell g + \sum_{k\le\ell} e^{it\sqrt{-\Delta}}P_k g \, e^{it\sqrt{-\Delta}}P_\ell g\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{3+1})}.
\end{aligned}$$

By the triangular inequality and symmetry

$$\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} \lesssim \Big(\sum_{\ell\ge 0}\|\sum_k e^{it\sqrt{-\Delta}}P_k g \, e^{it\sqrt{-\Delta}}P_{k+\ell}g\|_{L^2(\mathbb{R}^{3+1})}\Big)^{\frac{1}{2}}.$$

We observe now that for $\ell \ge 0$,

$$\begin{aligned}
\operatorname{supp}\Big((e^{it\sqrt{-\Delta}}P_k g \, e^{it\sqrt{-\Delta}}P_{k+\ell}g)^{\wedge x,t}\Big) &\subset \operatorname{supp}\Big(\widehat{P_k g} * \widehat{P_{k+\ell}g}\Big) \times \mathbb{R} \\
&\subset A_{k+\ell} \times \mathbb{R},
\end{aligned}$$

where $A_{k+\ell} = \mathcal{A}_{k+\ell-1} \cup \mathcal{A}_{k+\ell} \cup \mathcal{A}_{k+\ell+1}$.

Thus, the supports of the functions $\{\big(e^{it\sqrt{-\Delta}}P_k g \, e^{it\sqrt{-\Delta}}P_{k+\ell}g\big)^{\wedge x,t}\}_k$ are almost disjoint and therefore by L^2 orthogonality we have

$$\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} \lesssim \Big(\sum_{\ell>0}(\sum_k \|e^{it\sqrt{-\Delta}}P_k g \, e^{it\sqrt{-\Delta}}P_{k+\ell}g\|^2_{L^2(\mathbb{R}^{3+1})})^{\frac{1}{2}}\Big)^{\frac{1}{2}}. \tag{1.8}$$

Now we use a Whitney decomposition, in the spirit of [76] and [84] (see also [77] and [78]). For fixed k and $k+\ell$, let $\Gamma = \{(x,y) \in \mathcal{A}_k \times \mathcal{A}_{k+\ell} : \angle(x,y) = 0\}$. We decompose $\mathcal{A}_k \times \mathcal{A}_{k+\ell} \setminus \Gamma$ as follows: For every $j \in \mathbb{N}$, we decompose $\mathcal{A}_k$ and $\mathcal{A}_{k+\ell}$ in the sectors

$\tau_m^{j,k}$ and $\tau_m^{j,k+\ell}$ respectively. We say that $\tau_m^{j,k}$ is the parent of $\tau_m^{j+1,k}$ if $\tau_m^{j+1,k} \subset \tau_m^{j,k}$, and we write $\tau_m^{j,k} \sim \tau_{m'}^{j,k+\ell}$ if $\tau_m^{j,k+\ell}$ and $\tau_{m'}^{j,k+\ell}$ are not adjacent but have adjacent parents.

We write

$$\begin{aligned}
&\|e^{it\sqrt{-\Delta}}P_k g\; e^{it\sqrt{-\Delta}}P_{k+\ell}g\|_{L^2(\mathbb{R}^{3+1})}^2 \\
&\qquad = \|\sum_{j\geq 0}\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|_{L^2(\mathbb{R}^{3+1})}^2.
\end{aligned}$$

Lemma 1.1.

$$\begin{aligned}
&\|\sum_{j}\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|_{L^2(\mathbb{R}^{3+1})}^2 \\
&\qquad \lesssim \sum_{j}\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|_{L^2(\mathbb{R}^{3+1})}^2.
\end{aligned}$$

Proof. We again want to use orthogonality on the Fourier side. We claim that if $\tau_m^{j,k} \sim \tau_{m'}^{j,k+\ell}$,

$$\operatorname{supp}\Big((e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j)^{\wedge_{x,t}}\Big) \subset \tilde{\tau}_m^{j,k} + \tilde{\tau}_{m'}^{j,k+\ell} \subset H_m^{j,k}, \tag{1.9}$$

where by writing $d\big((\xi,\tau),\mathbf{C}\big) := |\tau - |\xi||$,

$$H_m^{j,k} := \left\{(\xi,\tau)\in\mathbb{R}^d\times\mathbb{R} :\quad d\big((\xi,\tau),\mathbf{C}\big)\sim 2^{-2j}2^k,\quad \angle(w_m,\xi)\lesssim 2^{-j}\right\}.$$

To see this, let $(y,|y|)\in\tilde{\tau}_m^{j,k}$, $(z,|z|)\in\tilde{\tau}_{m'}^{j,k+\ell}$, then

$$\begin{aligned}
d((y,|y|)+(z,|z|),\mathbf{C}) = |y|+|z| - |\,y+z| &= \frac{(|y|+|z|)^2 - |y+z|^2}{|y|+|z|+|z+y|} \\
&\sim \frac{2(|y||z| - y\cdot z)}{|y|+|z|} = \frac{2|y||z|\big(1-\frac{y\cdot z}{|y||z|}\big)}{|y|+|z|} \\
&= \frac{2|y||z|\big(1-\cos(\angle(y,z))\big)}{|y|+|z|} \sim \frac{|y||z|\angle(y,z)^2}{|y|+|z|} \\
&\sim 2^k 2^{-2j}.
\end{aligned}$$

On the other hand, as $\angle(w_m,w_{m'})\sim 2^{-j}$, we have

$$\angle(y+z,w_m)\leq \angle(y,w_m)+\angle(z,w_m)\lesssim 2^{-j}.$$

This concludes the proof of (1.9). As the cardinal of indices m' related with m is of order $O(1)$ and the sets $\{H_m^{j,k}\}_{j,m}$ are almost disjoint, we get the lemma by Plancherel's theorem and almost orthogonality.

□

Therefore, combining Lemma 1.1 and Corollary 1.1,

$$\begin{aligned}&\|e^{it\sqrt{-\Delta}}P_kg\ e^{it\sqrt{-\Delta}}P_{k+\ell}g\|^2_{L^2(\mathbb{R}^{3+1})}\\&\qquad\lesssim\ 2^{\ell(1-\frac{r_1}{2}+\epsilon)}\sum_j\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}}2^{2k}2^{k(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_kg_m^j}\|^2_{\frac{4}{4-r_1}}\|\widehat{P_{k+\ell}g_{m'}^j}\|^2_{\frac{4}{4-r_1}},\end{aligned}$$

for all $\frac{3}{2}\leq r_1\leq 2$.

Now, as the number of indices m' related with m is $O(1)$, using $2ab\leq\varepsilon a^2+\frac{1}{\varepsilon}b^2$ for $\varepsilon>0$,

$$\begin{aligned}&\|e^{it\sqrt{-\Delta}}P_kg\ e^{it\sqrt{-\Delta}}P_{k+\ell}g\|^2_{L^2(\mathbb{R}^{3+1})}\\&\qquad\lesssim 2^{\ell(1-\frac{r_1}{2}+\epsilon)}\Big(\sum_j\sum_m 2^{-\ell(1+\frac{(3r_1-6)}{2})}2^{2k}2^{k(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_kg_m^j}\|^4_{\frac{4}{4-r_1}}\\&\qquad\qquad+\sum_j\sum_{m'}2^{\ell(1+\frac{(3r_1-6)}{2})}2^{2k}2^{k(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^4_{\frac{4}{4-r_1}}\Big)\\&\qquad= 2^{-\ell(\frac{r_1}{2}+\frac{(3r_1-6)}{2}-\epsilon)}\Big(\sum_j\sum_m 2^{2k}2^{k(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_kg_m^j}\|^4_{\frac{4}{4-r_1}}\\&\qquad\qquad+\sum_j\sum_{m'}2^{2(k+\ell)}2^{(k+\ell)(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^4_{\frac{4}{4-r_1}}\Big).\end{aligned}$$

Inserting this into the estimate (1.8), we see that $\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})}$ is dominated by a constant times

$$\begin{aligned}&\Big(\sum_{\ell\geq 0}2^{-\frac{\ell}{2}(\frac{r_1}{2}+\frac{(3r_1-6)}{2}-\epsilon)}\Big(\sum_k\sum_j\sum_m 2^{2k}2^{k(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_kg_m^j}\|^4_{\frac{4}{4-r_1}}\\&\qquad+\sum_k\sum_j\sum_{m'}2^{2(k+\ell)}2^{(k+\ell)(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^4_{\frac{4}{4-r_1}}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}.\end{aligned}$$

Using the change of variables $k'=k+\ell$ for the second term

$$\begin{aligned}&\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})}\\&\qquad\lesssim\Big(\sum_{\ell\geq 0}2^{-\frac{\ell}{2}(\frac{r_1}{2}+\frac{(3r_1-6)}{2}-\epsilon)}\Big(\sum_k 2^{2k}\sum_j\sum_m 2^{k(3r_1-6)}2^{j(4-2r_1)}\|\widehat{P_kg_m^j}\|^4_{\frac{4}{4-r_1}}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}.\end{aligned}$$

Setting $p = \frac{4}{4-r_1}$, this is

$$\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} \lesssim \Big(\sum_{\ell\geq 0} 2^{-\frac{\ell}{2}(\frac{r_1}{2}+\frac{(3r_1-6)}{2}-\epsilon)}(\sum_k 2^{2k}\sum_j\sum_m |\tau_m^{j,k}|^{2\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^4)^{\frac{1}{2}}\Big)^{\frac{1}{2}}.$$

Now, for all $\dfrac{3}{2} < r_1 \leq 2$ (this implies $2 \geq p > \frac{8}{5}$), we can sum in ℓ, which yields

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} &\lesssim \Big(\sum_k 2^{2k}\sum_j\sum_m |\tau_m^{j,k}|^{2\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^4)^{\frac{1}{4}} \\ &= (\sum_k 2^{2k}\|\widehat{P_k g}\|^4_{X^k_{p,4}}\Big)^{\frac{1}{4}}.\end{aligned}$$

This concludes the proof of the first inequality. For the second inequality, by a simple adaptation of Theorem 1.3 in [3] or the forthcoming Lemma 1.3, we have for every $0 \leq \theta < \frac{1}{2}$ and $p < 2$,

$$(\sum_k 2^{2k}\|\widehat{P_k g}\|^4_{X^k_{p,4}})^{\frac{1}{4}} \lesssim \Big(\sum_k 2^{2k}\sup_{j,m}|\tau_m^{j,k}|^{2\theta\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^{4\theta}\|\widehat{P_k g}\|_2^{4(1-\theta)}\Big)^{\frac{1}{4}}. \tag{1.10}$$

Thus, taking a supremum in k,

$$\|e^{it\sqrt{-\Delta}}g\|_{L^4(\mathbb{R}^{3+1})} \lesssim \sup_{j,k,m} 2^{k\frac{\theta}{2}}|\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^{\theta}\|g\|^{1-\theta}_{\dot{B}^{\frac{1}{2}}_{2,4(1-\theta)}},$$

and we are done. We notice that by Hölder inequality in (1.10) we can get (1.2).

□

1.4. Proof of Theorem 1.3

In dimensions $d = 2$ and $d \geq 4$, we will need some additional lemmas. We begin by proving an easy generalization of Lemma 6.1 in [76], which is a cheap substitute for L^2 orthogonality in L^p. We present this generalization because in our case we will be working with functions Fourier supported in neighborhoods of the cone, instead of rectangles.

Lemma 1.2. *Let $(E_k)_{k\in\mathbb{Z}}$ be a collection of sets such that there exist almost disjoint $(F_k)_{k\in\mathbb{Z}}$, with $E_k \subset F_k$ for every k, such that there exist bump functions ϕ_{E_k} equal to 1 on E_k and 0 outside F_k, and such that*

$$\int |\widehat{\phi_{E_k}}(\xi)|d\xi \leq C \tag{1.11}$$

uniformly in k. Suppose that $(f_k)_{k\in\mathbb{Z}}$ are a collection of functions whose Fourier transforms are supported on $(E_k)_{k\in\mathbb{Z}}$. Then for all $1 \le p \le \infty$, we have

$$\|\sum_k f_k\|_p \lesssim C^{1-\frac{2}{p^*}} \left(\sum_k \|f_k\|_p^{p_*}\right)^{\frac{1}{p_*}}$$

where $p_* = \min(p, p')$ and $p^* = \max(p, p')$.

Proof. Let $\widehat{m_k f} = \phi_{E_k}\widehat{f}$. It will be enough to prove

$$\|\sum_k m_k g_k\|_p \lesssim C^{1-\frac{2}{p^*}} \left(\sum_k \|g_k\|_p^{p_*}\right)^{\frac{1}{p_*}}$$

for general functions g_k. The result then follows by taking $f_k = g_k = m_k g_k$. By interpolation it suffices to prove the inequality for the values $p = 1$, $p = 2$, $p = \infty$. The case $p = 2$ follows by Plancherel and using that the collection is almost disjoint. For $p = 1$, we note that

$$\begin{aligned}\|\sum_k m_k g_k\|_1 &\lesssim \sum_k \|m_k g_k\|_1 = \sum_k \|\widehat{\phi_{E_k}} * g_k\|_1 \\ &\le \sum_k \|\widehat{\phi_{E_k}}\|_1 \|g_k\|_1 \le C \sum_k \|g_k\|_1.\end{aligned}$$

Similarly for $p = \infty$

$$\begin{aligned}\|\sum_k m_k g_k\|_\infty &\le \sum_k \|m_k g_k\|_\infty = \sum_k \|\widehat{\phi_{E_k}} * g_k\|_\infty \\ &\le \sum_k \|\widehat{\phi_{E_k}}\|_1 \|g_k\|_\infty \le C \sum_k \|g_k\|_\infty.\end{aligned}$$

□

Remark 1.2. *The standard case is when $F_k = (1 + c)(E_k - \mathbf{c}(E_k)) + \mathbf{c}(E_k)$ for some $c > 0$ and $\{E_k\}_k$ are rectangles. Here $\mathbf{c}(E_k)$ is the centre of E_k, so this is nothing more than a slightly larger rectangle with the same centre. The condition (1.11) is then satisfied with $C = C(d)$.*

The next lemma refines the well known embedding $L^2 \hookrightarrow X^0_{p,q}$ (see [8], [3], [67]).

Lemma 1.3. *Let $q > 2$, and $1 < p < 2$. Then*

$$\sum_j \Big(\sum_m |\tau_m^{j,k}|^{q\frac{p-2}{2p}} \|\widehat{P_k g_m^j}\|_p^q\Big)^{\frac{2}{q}} \lesssim \|P_k g\|_2^2.$$

The key ingredient in the proof is an atomic decomposition of L^p due to Keel and Tao [39].

Lemma 1.4. [39] *Let $f \in L^p(\mathbb{R}^d)$ for some $1 < p < \infty$. Then, we can decompose*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \chi_n(x),$$

where χ_n are functions bounded in magnitude by 1 and supported in disjoint sets of measure at most 2^n, and c_n are non-negative real numbers such that

$$\sum_{n\in\mathbb{Z}} 2^n |c_n|^p \sim \|f\|_p^p.$$

We also need a simple inequality used in [76], which allows us to get some gain when we sum over a partition in norm ℓ^p, for $p \geq 1$. It follows easily from the cases $p = 1$ and $p = \infty$.

Lemma 1.5. [76] *Let $p \geq 1$, then*

$$\sum_m |\Omega \cap \tau_m^{j,k}|^p \lesssim |\Omega| \min(|\Omega|, |\tau_m^{j,k}|)^{p-1}.$$

Proof of Lemma 1.3. Using Lemma 1.4, we can decompose

$$\widehat{P_k g} = \sum_n c_n \chi_n,$$

where the χ_n have disjoint supports, H_n, with $|H_n| \leq 2^n$ and

$$\sum_n 2^n |c_n|^2 \sim \|P_k g\|_2^2. \tag{1.12}$$

Using that H_n have disjoint supports

$$\begin{aligned}(*) := \sum_j |\tau_m^{j,k}|^{\frac{p-2}{p}} \Big(\sum_m \|\widehat{P_k g_m^j}\|_p^q\Big)^{\frac{2}{q}} &\lesssim \sum_j |\tau_m^{j,k}|^{\frac{p-2}{p}} \Big(\sum_m \Big(\int_{\tau_m^{j,k}} \sum_n |c_n \chi_{H_n}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{2}{q}} \\ &\lesssim \sum_j |\tau_m^{j,k}|^{\frac{p-2}{p}} \Big(\sum_m \Big(\sum_n |c_n|^p |H_n \cap \tau_m^{j,k}|\Big)^{\frac{q}{p}}\Big)^{\frac{2}{q}}.\end{aligned}$$

By Minkowski's inequality and the hypothesis $\frac{q}{p} > 1$,

$$(*) \lesssim \sum_j |\tau_m^{j,k}|^{\frac{p-2}{p}} \Big(\sum_n |c_n|^p \Big(\sum_m |H_n \cap \tau_m^{j,k}|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big)^{\frac{2}{p}}.$$

We split the sum in n, and use Lemma 1.5, so that

$$(*) \lesssim \sum_j |\tau_m^{j,k}|^{\frac{p-2}{p}} \Big(\sum_{n>kd-j(d-1)} |c_n|^p (\sum_m |H_n \cap \tau_m^{j,k}|^{\frac{q}{p}})^{\frac{p}{q}} + \sum_{n\le kd-j(d-1)} |c_n|^p (\sum_m |H_n \cap \tau_m^{j,k}|^{\frac{q}{p}})^{\frac{p}{q}} \Big)^{\frac{2}{p}}$$
$$\lesssim \sum_j |\tau_m^{j,k}|^{\frac{p-2}{p}} \Big(\sum_{n>kd-j(d-1)} |c_n|^p |\tau_m^{j,k}|^{\frac{q-p}{q}} 2^{n\frac{p}{q}} + \sum_{n\le kd-j(d-1)} |c_n|^p 2^n \Big)^{\frac{2}{p}}.$$

Simplifying,

$$(*) \lesssim \sum_j \Big(\sum_{n>kd-j(d-1)} |c_n|^p |\tau_m^{j,k}|^{\frac{p}{2}-\frac{p}{q}} 2^{n\frac{p}{q}} + \sum_{n\le kd-j(d-1)} |c_n|^p 2^{n\frac{p}{2}} 2^{n(1-\frac{p}{2})} |\tau_m^{j,k}|^{\frac{p}{2}-1} \Big)^{\frac{2}{p}}$$
$$\lesssim \sum_j \Big(\sum_{n>kd-j(d-1)} |c_n|^p 2^{n\frac{p}{2}} 2^{(\frac{p}{2}-\frac{p}{q})(kd-j(d-1)-n)} + \sum_{n\le kd-j(d-1)} |c_n|^p 2^{n\frac{p}{2}} 2^{(\frac{p}{2}-1)(kd-j(d-1)-n)} \Big)^{\frac{2}{p}}.$$

As $p<2$, by Hölder's inequality,

$$(*) \lesssim \sum_j \Big(\sum_{n>kd-j(d-1)} |c_n|^2 2^n 2^{\frac{1}{p}(\frac{p}{2}-\frac{p}{q})(kd-j(d-1)-n)} \big(\sum_{n>kd-j(d-1)} 2^{\frac{1}{2-p}(\frac{p}{2}-\frac{p}{q})(kd-j(d-1)-n)} \big)^{\frac{2-p}{p}}$$
$$+ \sum_{n\le kd-j(d-1)} |c_n|^2 2^n 2^{\frac{1}{p}(\frac{p}{2}-1)(kd-j(d-1)-n)} \big(\sum_{n\le kd-j(d-1)} 2^n 2^{\frac{1}{2}(n-(kd-j(d-1)))} \big)^{\frac{2-p}{p}} \Big).$$

Again as $p<2$ and $q>2$ we can sum, so that

$$(*) \lesssim \sum_j \Big(\sum_{n>kd-j(d-1)} |c_n|^2 2^n 2^{\frac{1}{p}(\frac{p}{2}-\frac{p}{q})(kd-j(d-1)-n)} + \sum_{n\le kd-j(d-1)} |c_n|^2 2^n 2^{\frac{1}{p}(\frac{p}{2}-1)(kd-j(d-1)-n)} \Big)$$
$$\lesssim \sum_n |c_n|^2 2^n \sum_{(d-1)j\ge kd-n} 2^{\frac{1}{p}(\frac{p}{2}-\frac{p}{q})(kd-j(d-1)-n)} + \sum_n |c_n|^2 2^n \sum_{0\le (d-1)j\le kd-n} 2^{\frac{1}{p}(\frac{p}{2}-1)(kd-j(d-1)-n)}$$
$$\lesssim \sum_n |c_n|^2 2^n.$$

So we conclude the result using (1.12).

□

Proof of Theorem 1.3. By the triangle inequality and symmetry as before,

$$\|e^{it\sqrt{-\Delta}} g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Big(\sum_{\ell>0} \| \sum_k e^{it\sqrt{-\Delta}} P_k g \; e^{it\sqrt{-\Delta}} P_{k+\ell} g \|_{L^r(\mathbb{R}^{d+1})} \Big)^{\frac{1}{2}},$$

where $r = \frac{q}{2} = \frac{d+1}{d-1}$. We use again that,

$$\operatorname{supp}\Big((e^{it\sqrt{-\Delta}}P_k g\; e^{it\sqrt{-\Delta}}P_{k+\ell}g)^{\wedge_{x,t}}\Big) \subset A_{k+\ell}\times\mathbb{R},$$

however, we are no longer in L^2, and so, instead we apply Lemma 1.2. We cover $A_{k+\ell}$ by a finite collection of rectangles $\{R_{k,n}\}_n$ of cardinality depending on the dimension, which are at a distance $\sim 2^{k+\ell}$ to the origin. We set $E_{k,n} = R_{k,n}\times\mathbb{R}$ and we have by construction that for some small $c>0$, the sets $F_{k,n} = (1+c)(E_{k,n}-\mathbf{c}(E_{k,n}))+\mathbf{c}(E_{k,n})$ are almost disjoint. Thus, the hypothesis of Lemma 1.2 are satisfied, so that

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Big(\sum_{\ell>0}\Big(\sum_k \|e^{it\sqrt{-\Delta}}P_k g\; e^{it\sqrt{-\Delta}}P_{k+\ell}g\|^{r_*}_{L^r(\mathbb{R}^{d+1})}\Big)^{\frac{1}{r_*}}\Big)^{\frac12}, \tag{1.13}$$

where $r_* = \min(r, r')$. That is, $r_* = r$ if $d\geq 3$ and $r_* = r'$ if $d=2$. As before we use the Whitney decomposition

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}P_k g\; e^{it\sqrt{-\Delta}}P_{k+\ell}g\|^{r_*}_{L^r(\mathbb{R}^{d+1})}\\ = \|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|^{r_*}_{L^r(\mathbb{R}^{d+1})}.\end{aligned}$$

Again, we have to deal with orthogonality in L^p. We need the following lemmas.

Lemma 1.6.

$$\begin{aligned}&\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|^{r_*}_{L^r(\mathbb{R}^{d+1})}\\ &\qquad\lesssim \Big(\sum_j\Big(\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|^{r_*}_{L^r(\mathbb{R}^{d+1})}\Big)^{\frac{1}{r_*}}\Big)^{r_*}.\end{aligned}$$

Proof. By the triangle inequality, it will suffice to prove for fixed j, k and ℓ,

$$\begin{aligned}&\|\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|^{r_*}_{L^r(\mathbb{R}^{d+1})}\\ &\qquad\lesssim \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|^{r_*}_{L^r(\mathbb{R}^{d+1})}.\end{aligned}$$

This will follow from Lemma 1.2. Indeed, we have the set inclusion

$$\operatorname{supp}\Big((e^{it\sqrt{-\Delta}}P_k g_m^j\; e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j)^{\wedge_{x,t}}\Big) \subset (\tau_m^{j,k}+\tau_{m'}^{j,k+\ell})\times\mathbb{R}.$$

Fix j and observe that $(\tau_{m'}^{j,k+\ell}+\tau_m^{j,k})\times\mathbb{R}\subset 4(\tau_{m'}^{j,k+\ell}-\mathbf{c}(\tau_{m'}^{j,k+\ell}))+\mathbf{c}(\tau_{m'}^{j,k+\ell})\times\mathbb{R}=E_{m'}$. Now, as $F_{m'}=((1+c)(E_{m'}-\mathbf{c}(E_{m'}))+\mathbf{c}(E_{m'}))\times\mathbb{R}$ are almost disjoint for small $c>0$ and the cardinality of the indices m related with m' is of order $O(1)$, we can use Lemma 1.2 with $C=C(d)$, as every set $E_{m'}$ after a rotation is a dilation of $\tau_{m_0}^{1,1}$ for some m_0, to conclude the proof.

□

Lemma 1.7.

$$\begin{aligned}\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} & e^{it\sqrt{-\Delta}}P_k g_m^j\, e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}\\ &\lesssim 2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})}\sum_j\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}}\|e^{it\sqrt{-\Delta}}P_k g_m^j\, e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}.\end{aligned}$$

Proof. As before we want to use Lemma 1.2, but this time including the summation in j. As in the proof of Lemma 1.1, we have

$$\operatorname{supp}\Big((e^{it\sqrt{-\Delta}}P_k g_m^j\, e^{it\sqrt{-\Delta}}P_{k+\ell}g_{m'}^j)^{\wedge_{x,t}}\Big)\subset\tilde{\tau}_m^{j,k}+\tilde{\tau}_{m'}^{j,k+\ell}\subset H_m^{j,k,\ell},$$

where

$$H_m^{j,k,\ell}:=\left\{(\xi,\tau)\in A_{k+\ell}\times\mathbb{R}:\quad d\big((\xi,\tau),\mathbf{C}\big)\sim 2^{-2j}2^k,\quad \angle(w_m,\xi)\lesssim 2^{-j}\right\}.$$

Let $\tilde{H}_m^{j,k,\ell}$ be the $2^{-2j}2^k$ neighborhood of $H_m^{j,k,\ell-1}\cup H_m^{j,k,\ell}\cup H_m^{j,k,\ell+1}$, and let $\phi_{H_m^{j,k,\ell}}$ be a bump function which is 1 on $H_m^{j,k,\ell}$ and 0 outside $\tilde{H}_m^{j,k,\ell}$. We will show that we can find such functions $\phi_{H_m^{j,k,\ell}}$ with

$$\int|\widehat{\phi_{H_m^{j,k,\ell}}}(\xi)|d\xi\le C(d)2^{\ell\frac{(d-1)}{2}}$$

uniformly in j,m. As the sets $\{\tilde{H}_m^{j,k,\ell}\}_{j,m}$ are almost disjoints, we can apply Lemma 1.2 to get the result. To show that we can find these functions, we decompose $H_m^{j,k,\ell}$ in the sets

$$H_{m,\theta}^{j,k,\ell}:=H_m^{j,k,\ell}\cap\{(\xi,\tau)\in\mathbb{R}^{d+1}:\angle(w_\theta,\xi)\lesssim 2^{-j}2^{-\frac{\ell}{2}}\},$$

where $\{w_\theta\}\subset S^{d-1}$ is a maximally $2^{-j}2^{-\frac{\ell}{2}}$-separated grid such that $\angle(w_\theta,w_m)\lesssim 2^{-j}$. This set has cardinality $\lesssim 2^{\ell\frac{(d-1)}{2}}$. The key point is that we can find rectangles $R_{m,\theta}^{j,k,\ell}$ such that $H_{m,\theta}^{j,k,\ell}\subset R_{m,\theta}^{j,k,\ell}$ and $|H_{m,\theta}^{j,k,\ell}|\sim|R_{m,\theta}^{j,k,\ell}|$.

Let $\phi_{H_{m,\theta}^{j,k,\ell}}$ be a bump function which is equal to 1 on some rectangle $\tilde{R}_{m,\theta}^{j,k,\ell}$ which is contained in $R_{m,\theta}^{j,k,\ell}$, and is 0 in $(1+c)(\tilde{R}_{m,\theta}^{j,k,\ell}-\mathbf{c}(\tilde{R}_{m,\theta}^{j,k,\ell}))+\mathbf{c}(\tilde{R}_{m,\theta}^{j,k,\ell})$ for some $c>0$. We have then $\|\widehat{\phi_{H^{j,k,\ell}_{m,\theta}}}\|_{L^1} \lesssim 1$ uniformly in j,m. Therefore for a correct choice of $\{\tilde{R}^{j,k,\ell}_{m,\theta}\}_\theta$ we can set $\phi_{H^{j,k,\ell}_m} = \sum_\theta \phi_{H^{j,k,\ell}_{m,\theta}}$ satisfying the required properties.

$\square$

We will require both Lemma 1.6 and Lemma 1.7 in order to obtain the refinement. Lemma 1.6 alone is not sufficient due to the power of $\frac{1}{r_*}$ that appears. On the other hand, Lemma 1.7 is not sufficient as the constant $2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})}$ does not permit to sum in ℓ. In order to take advantage of the positive aspects of both lemmas we introduce $r_2 < r_*$ to be determined later (see (1.16) and (1.17)). We obtain

$$
\begin{aligned}
(*) := \|\sum_j \sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} & e^{it\sqrt{-\Delta}} P_k g^j_m \; e^{it\sqrt{-\Delta}} P_{k+\ell} g^j_{m'} \|^{r_*}_{L^r(\mathbb{R}^{d+1})} \\
= \|\sum_j \sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} & e^{it\sqrt{-\Delta}} P_k g^j_m \; e^{it\sqrt{-\Delta}} P_{k+\ell} g^j_{m'} \|^{r_2}_{L^r(\mathbb{R}^{d+1})} \\
\times \|\sum_j \sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} & e^{it\sqrt{-\Delta}} P_k g^j_m \; e^{it\sqrt{-\Delta}} P_{k+\ell} g^j_{m'} \|^{r_*-r_2}_{L^r(\mathbb{R}^{d+1})} \\
\lesssim \Big(\sum_j \big(\sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} & \|e^{it\sqrt{-\Delta}} P_k g^j_m \; e^{it\sqrt{-\Delta}} P_{k+\ell} g^j_{m'} \|^{r_*}_{L^r(\mathbb{R}^{d+1})} \big)^{\frac{1}{r_*}} \Big)^{r_2} \\
\times \Big(2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})} \sum_j \sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} & \|e^{it\sqrt{-\Delta}} P_k g^j_m \; e^{it\sqrt{-\Delta}} P_{k+\ell} g^j_{m'} \|^{r_*}_{L^r(\mathbb{R}^{d+1})} \Big)^{1-\frac{r_2}{r_*}}.
\end{aligned}
$$

Using Corollary 1.1, and writing $a_r = \frac{d-1}{2}(r_* - 2\frac{r_*}{r^*})\frac{r_*-r_2}{r_*}$, this is dominated by $2^{\ell a_r}$ times

$$
\begin{aligned}
&\Big(\sum_j \big(\sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} 2^{\ell\frac{r_*}{r}(1-\frac{r_1}{2}+\epsilon)} 2^{k\frac{r_*}{r}(r_1 d-(d+1))} 2^{j\frac{r_*}{r}(d+1-r_1(d-1))} \|\widehat{P_k g^j_m}\|^{r_*}_{\frac{2r}{2r-r_1}} \\
&\qquad\qquad \|\widehat{P_{k+\ell} g^j_{m'}}\|^{r_*}_{\frac{2r}{2r-r_1}} \big)^{\frac{1}{r_*}} \Big)^{r_2} \\
&\times \Big(\sum_j \sum_{m,m':\tau^{j,k}_m \sim \tau^{j,k+\ell}_{m'}} 2^{\ell\frac{r_*}{r}(1-\frac{r_1}{2}+\epsilon)} 2^{k\frac{r_*}{r}(r_1 d-(d+1))} 2^{j\frac{r_*}{r}(d+1-r_1(d-1))} \|\widehat{P_k g^j_m}\|^{r_*}_{\frac{2r}{2r-r_1}} \\
&\qquad\qquad \|\widehat{P_{k+\ell} g^j_{m'}}\|^{r_*}_{\frac{2r}{2r-r_1}} \Big)^{1-\frac{r_2}{r_*}}.
\end{aligned}
$$

where

$$\frac{5}{3} < r_1 < 2 \quad \text{for} \quad d = 2, \qquad \text{and} \qquad \frac{d+3}{d+1} < r_1 < \frac{d+1}{d-1} \quad \text{for} \quad d > 3. \tag{1.14}$$

Rewriting,

$$\begin{aligned}
(*) \lesssim\ & 2^{\ell \frac{r_*}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon)} \\
& \Big(\sum_j \big(\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} 2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_k g_m^j}\|^{r_*}_{\frac{2r}{2r-r_1}}\|\widehat{P_{k+\ell}g_{m'}^j}\|^{r_*}_{\frac{2r}{2r-r_1}}\big)^{\frac{1}{r_*}}\Big)^{r_2} \\
& \times\Big(\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} 2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_k g_m^j}\|^{r_*}_{\frac{2r}{2r-r_1}}\|\widehat{P_{k+\ell}g_{m'}^j}\|^{r_*}_{\frac{2r}{2r-r_1}}\Big)^{1-\frac{r_2}{r_*}}.
\end{aligned}$$

As the number of indices m' related with m is $O(1)$, using $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ for $\varepsilon > 0$,

$$\begin{aligned}
(*) \lesssim\ & 2^{\ell \frac{r_*}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon)} \\
& \Big(\sum_j \big(\sum_m 2^{-\ell\frac{r_*}{2}}2^{-\ell\frac{r_*}{r}\frac{(r_1-r)d}{2}}2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_k g_m^j}\|^{2r_*}_{\frac{2r}{2r-r_1}} \\
& \qquad + \sum_{m'} 2^{\ell\frac{r_*}{2}}2^{\ell\frac{r_*}{r}\frac{(r_1-r)d}{2}}2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^{2r_*}_{\frac{2r}{2r-r_1}}\big)^{\frac{1}{r_*}}\Big)^{r_2} \\
& \times\Big(\sum_j \sum_m 2^{-\ell\frac{r_*}{2}}2^{-\ell\frac{r_*}{r}\frac{(r_1-r)d}{2}}2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_k g_m^j}\|^{2r_*}_{\frac{2r}{2r-r_1}} \\
& \qquad + \sum_{m'} 2^{\ell\frac{r_*}{2}}2^{\ell\frac{r_*}{r}\frac{(r_1-r)d}{2}}2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^{2r_*}_{\frac{2r}{2r-r_1}}\Big)^{1-\frac{r_2}{r_*}}.
\end{aligned}$$

That is,

$$\begin{aligned}
(*) \lesssim\ & 2^{\ell \frac{r_*}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon-\frac{r}{2}-\frac{(r_1-r)d}{2})} \\
& \Big(\sum_j \big(\sum_m 2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_k g_m^j}\|^{2r_*}_{\frac{2r}{2r-r_1}} \\
& \qquad + \sum_{m'} 2^{(k+\ell)r_*}2^{(k+\ell)\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^{2r_*}_{\frac{2r}{2r-r_1}}\big)^{\frac{1}{r_*}}\Big)^{r_2} \\
& \times\Big(\sum_j \sum_m 2^{kr_*}2^{k\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_k g_m^j}\|^{2r_*}_{\frac{2r}{2r-r_1}} \\
& \qquad + \sum_{m'} 2^{(k+\ell)r_*}2^{(k+\ell)\frac{r_*}{r}(r_1-r)d}2^{j\frac{r_*}{r}(r-r_1)(d-1)}\|\widehat{P_{k+\ell}g_{m'}^j}\|^{2r_*}_{\frac{2r}{2r-r_1}}\Big)^{1-\frac{r_2}{r_*}}.
\end{aligned}$$

Inserting into the estimate (1.13) and writing $k' = k + \ell$, we get

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Bigg(\sum_{\ell \geq 0} 2^{\ell \frac{1}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon-\frac{r}{2}-\frac{(r_1-r)d}{2})} \Bigg(\sum_k \Big(\sum_j \big(\sum_m 2^{kr_*} 2^{k\frac{r_*}{r}(r_1-r)d} 2^{j\frac{r_*}{r}(r-r_1)(d-1)} \|\widehat{P_k g_m^j}\|_{\frac{2r}{2r-r_1}}^{2r_*} \big)^{\frac{1}{r_*}} \Big)^{r_2}$$

$$\times \big(\sum_j \sum_m 2^{kr_*} 2^{k\frac{r_*}{r}(r_1-r)d} 2^{j\frac{r_*}{r}(r-r_1)(d-1)} \|\widehat{P_k g_m^j}\|_{\frac{2r}{2r-r_1}}^{2r_*} \big)^{1-\frac{r_2}{r_*}} \Bigg)^{\frac{1}{r_*}} \Bigg)^{\frac{1}{2}}.$$

Setting $p = \frac{2r}{2r-r_1}$, we have that $\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})}$ is dominated by a constant multiple of

$$\Bigg(\sum_{\ell \geq 0} 2^{\ell \frac{1}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon-\frac{r}{2}-\frac{(r_1-r)d}{2})} \Bigg(\sum_k 2^{kr_*} \Big(\sum_j \big(\sum_m |\tau_m^{j,k}|^{r_*\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^{2r_*} \big)^{\frac{1}{r_*}} \Big)^{r_2}$$

$$\times \big(\sum_j \sum_m |\tau_m^{j,k}|^{r_*\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^{2r_*} \big)^{1-\frac{r_2}{r_*}} \Bigg)^{\frac{1}{r_*}} \Bigg)^{\frac{1}{2}}.$$

Taking suprema,

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Bigg(\sum_{\ell \geq 0} 2^{\ell \frac{1}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon-\frac{r}{2}-\frac{(r_1-r)d}{2})} \Bigg(\sum_k 2^{kc(r_*-r_2)} \sup_{j,m} |\tau_m^{j,k}|^{c(r_*-r_2)\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^{2c(r_*-r_2)}$$

$$\times 2^{k(r_*-c(r_*-r_2))} \Big(\sum_j \big(\sum_m |\tau_m^{j,k}|^{r_*\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^{2r_*} \big)^{\frac{1}{r_*}} \Big)^{r_2}$$

$$\big(\sum_j \sum_m |\tau_m^{j,k}|^{(1-c)r_*\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^{2(1-c)r_*} \big)^{1-\frac{r_2}{r_*}} \Bigg)^{\frac{1}{r_*}} \Bigg)^{\frac{1}{2}},$$

where $(1-c)r_* > 1$, that is $c < 1 - \frac{1}{r_*}$. Using Lemma 1.3,

(1.15)
$$\begin{aligned}&\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})}\\ &\quad\lesssim \Big(\sum_{\ell\geq 0} 2^{\ell\frac{1}{r}(1-\frac{r_1}{2}+a_r\frac{r}{r_*}+\epsilon-\frac{r}{2}-\frac{(r_1-r)d}{2})}\\ &\Big(\sum_k 2^{kc(r_*-r_2)}\sup_{j,m}|\tau_m^{j,k}|^{c(r_*-r_2)\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^{2c(r_*-r_2)}2^{k(r_*-c(r_*-r_2))}\|\widehat{P_k g}\|_2^{2r_*-2c(r_*-r_2)}\Big)^{\frac{1}{r_*}}\Big)^{\frac{1}{2}}.\end{aligned}$$

We want to be able to sum in ℓ, therefore we require

$$1-\frac{r_1}{2}+a_r\frac{r}{r_*}-\frac{r}{2}-\frac{(r_1-r)d}{2} < 0.$$

For the case $d=2$, this is insured by

$$r_2 > 9(1-\frac{r_1}{2}). \tag{1.16}$$

For the cases $d > 3$, we require instead that

$$r_2 > \frac{2d(d+1)}{(d-3)(d-1)} - r_1\frac{(d+1)^2}{(d-3)(d-1)}. \tag{1.17}$$

Thus, summing in ℓ and taking a supremum in k,

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} &\lesssim \sup_{j,m,k} 2^{k\frac{c}{2}(1-\frac{r_2}{r_*})}|\tau_m^{j,k}|^{\frac{c}{2}(1-\frac{r_2}{r_*})\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^{c(1-\frac{r_2}{r_*})}\\ &\qquad\Big(\sum_k 2^{k(r_*-c(r_*-r_2))}\|\widehat{P_k g}\|_2^{2r_*-2c(r_*-r_2)}\Big)^{\frac{1}{2r_*}}.\end{aligned}$$

This can be rewritten as,

$$\|e^{it\sqrt{-\Delta}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \sup_{j,m,k} 2^{k\frac{c}{2}(1-\frac{r_2}{r_*})}|\tau_m^{j,k}|^{\frac{c}{2}(1-\frac{r_2}{r_*})\frac{p-2}{p}}\|\widehat{P_k g_m^j}\|_p^{c(1-\frac{r_2}{r_*})}\|g\|_{\dot{B}^{\frac{1}{2}}_{2,2r_*(1-c(1-\frac{r_2}{r_*}))}}^{(1-c(1-\frac{r_2}{r_*}))},$$

which is the desired inequality. We set $\theta = c\dfrac{(r_*-r_2)}{r_*}$, where $0 \leq c < 1 - \frac{1}{r_*}$.

For $d=2$, to ensure (1.16) we take

$$0 \leq \theta < \frac{1}{3}\Big(1-\frac{2r_2}{3}\Big) < r_1 - \frac{5}{3}.$$

For $d > 3$, to ensure (1.17) we take

$$0 \leq \theta < \frac{2}{d+1}\Big(1 - \frac{(d-1)r_2}{d+1}\Big) < r_1\frac{2}{d-3} - \frac{2(d+3)}{(d-3)(d+1)},$$

and we are done. We notice that by Hölder inequality in (1.15) we can get (1.2)

$\square$

CHAPTER 2

The linear profile decomposition

2.1. Introduction

We consider now the wave equation with general initial data $u(\cdot,0)=u_0$, $\partial_t u(\cdot,0)=u_1$. Its solution can be written as

(2.1)

$$\begin{aligned} u(\cdot,t) &= S[u_0,u_1](\cdot,t) = S_+[u_0,u_1](\cdot,t) + S_-[u_0,u_1](\cdot,t) \\ &= \frac{1}{2}\big(e^{it\sqrt{-\triangle}}u_0 + \frac{1}{i}\frac{e^{it\sqrt{-\triangle}}u_1}{\sqrt{-\triangle}}\big) + \frac{1}{2}\big(e^{-it\sqrt{-\triangle}}u_0 - \frac{1}{i}\frac{e^{-it\sqrt{-\triangle}}u_1}{\sqrt{-\triangle}}\big), \end{aligned}$$

where $\widehat{\sqrt{-\triangle}f}(\xi) = |\xi|\widehat{f}(\xi)$. An easy consequence of the Strichartz inequality (1.1) is that

$$\|S[u_0,u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \le C\|(u_0,u_1)\|_{\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)}, \tag{2.2}$$

where $\|(u_0,u_1)\|_{\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)}$ is the norm in the product Sobolev space $\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)$ defined as

$$\|(u_0,u_1)\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)} = \|u_0\|^2_{\dot H^{\frac12}(\mathbb{R}^d)} + \|u_1\|^2_{\dot H^{-\frac12}(\mathbb{R}^d)}.$$

For $(u_0,u_1)\in \dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)$, the energy $E(u_0,u_1) = \|(u_0,u_1)\|_{\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)}$ is conserved for solutions of (2.1), that is, for all $t\in\mathbb{R}$ we have $E(u(t),\partial_t u(t)) = E(u_0,u_1)$.

Theorem 1.1 will enable us to prove a profile decomposition for the wave equation with initial data in $\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)$ for dimensions $d\ge 2$. Similar decompositions were obtained previously by Bahouri and Gérard [1] with initial data in $\dot H^1\times L^2(\mathbb{R}^3)$, and Bulut [13] with initial data in $\dot H^s\times \dot H^{s-1}(\mathbb{R}^d)$ for $d\ge 3$ and $s\ge 1$. For profile decompositions for the Schrödinger equation see [3], [14], [45], [58], [69], for the Klein–Gordon equation see [48], and for a large class of dispersive propagator see [29].

We need to introduce some definitions in order to state the profile decomposition. For a bounded sequence $(\mathbf{u_0},\mathbf{u_1}) = (u_{0,n},u_{1,n})_n$ in $\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)$ we define the value

$$\|(\mathbf{u_0},\mathbf{u_1})\| = \sup_n \|(u_{0,n},u_{1,n})\|_{\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)}.$$

If $\{(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{n\in\mathbb{N}}\}_{j\in\mathbb{N}}$ is a family of sequences in $\mathbb{R}^+\setminus\{0\}\times[1,\infty)\times\mathbb{S}^{d-1}\times\mathbb{R}^d\times\mathbb{R}$ and $T_{w_j^n}^{\ell_j^n}$ is the rescaled Lorentz transformation defined in (1.7), then we say that the family is orthogonal if one of the following properties is satisfied for all $j \neq k$:

A. Lorentz property

$$\frac{\ell_j^n}{\ell_k^n} + \frac{\ell_k^n}{\ell_j^n} \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.3}$$

B. Rescaling property

$$\frac{r_j^n}{r_k^n} + \frac{r_k^n}{r_j^n} \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.4}$$

C. Angular property

$$r_j^n \sim r_k^n,\ \ell_j^n \sim \ell_k^n \quad \text{and} \quad \ell_j^n |w_j^n - w_k^n| \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.5}$$

D. Space-time translation property

$$r_j^n = r_k^n,\ \ell_j^n = \ell_k^n,\ w_j^n = w_k^n \quad \text{and} \quad \left|(T_{w_j^n}^{\ell_j^n})^{-1} r_j^n (x_j^n - x_k^n, t_j^n - t_k^n)\right| \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.6}$$

For each $(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n) \in \mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, we define the transformations Γ_j^n by

$$\Gamma_j^n F(x,t) = \left(\frac{r_j^n}{\ell_j^n}\right)^{\frac{d-1}{2}} F\Big((T_{w_j^n}^{\ell_j^n})^{-1} r_j^n (x - x_j^n, t - t_j^n)\Big).$$

These transformations conserve the $L^{2\frac{d+1}{d-1}}$ norm, that is

$$\|\Gamma_j^n F\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = \|F\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.$$

The importance of the orthogonality of the sequences becomes clear in the following lemmas, which will be proved in Section 2.4.

Lemma 2.1. *Let $d \geq 2$, $\{(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{n\in\mathbb{N}}\}_{1\leq j\leq N}$ in $\mathbb{R}^+ \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, be an orthogonal family of sequences, and $\{S[\phi_0^j, \phi_1^j]\}_{1\leq j\leq N}$ be a sequence of functions in $L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})$. Then for every $N \geq 1$ we have*

$$\limsup_{n\to\infty} \|\sum_{j=1}^N \Gamma_j^n S[\phi_0^j, \phi_1^j]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} = \sum_{j=1}^N \|S[\phi_0^j, \phi_1^j]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}}.$$

Lemma 2.2. *Let $d \geq 2$, $\{(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{n\in\mathbb{N}}\}_{1\leq j\leq 2}$ in $\mathbb{R}^+ \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, be two orthogonal sequences, and $\{S[\phi_0^1, \phi_1^1]\}$ be a function in $L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})$. Then we have*

$$(\Gamma_2^n)^{-1}\Gamma_1^n S[\phi_0^1, \phi_1^1] \underset{n\to\infty}{\rightharpoonup} 0 \qquad \textit{weakly in } L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1}).$$

The following theorem is the main result of this chapter.

Theorem 2.1. *Let $(u_{0,n}, u_{1,n})_n$ be a bounded sequence in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ with $d \geq 2$. Then, there exist a subsequence (still denoted $(u_{0,n}, u_{1,n})_n$), a sequence $(\phi_0^j, \phi_1^j)_{j\in\mathbb{N}} \subset \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ and a family of orthogonal sequences $\{(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{n\in\mathbb{N}}\}_{j\in\mathbb{N}}$ in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, such that for every $N \geq 1$,*

$$S[u_{0,n}, u_{1,n}](x,t) = \sum_{j=1}^{N} \Gamma_j^n S[\phi_0^j, \phi_1^j](x,t) + S[R_{0,n}^N, R_{1,n}^N](x,t), \tag{2.7}$$

with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S[R_{0,n}^N, R_{1,n}^N]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = 0. \tag{2.8}$$

Furthermore, we also have for every $N \geq 1$,

(2.9)

$$\|(u_{0,n}, u_{1,n})\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} = \sum_{j=1}^{N} \|(\phi_0^j, \phi_1^j)\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + \|(R_{0,n}^N, R_{1,n}^N)\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + o(1), \quad n\to\infty.$$

The existence of maximizers for the Strichartz inequality, is an easy consequence of the profile decomposition. For progress on closely related problems see [6], [13], [15], [18, 19], [26], [27], [28], [36], [56], [62], [63], [68] and [69].

Corollary 2.1. *Let $d \geq 2$, then there exists a maximizing pair $(\psi_0, \psi_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ such that*

$$\|S[\psi_0, \psi_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = W(d)\|(\psi_0, \psi_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)},$$

where

$$W(d) := \sup\Big\{\|S[\phi_0, \phi_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \colon (\phi_0, \phi_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} \\ \textit{with } \|(\phi_0, \phi_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} = 1\Big\}.$$

Proof. We choose $(u_{0,n}, u_{1,n}) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ such that $\|(u_{0,n}, u_{1,n})\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} = 1$ and $\|S[u_{0,n}, u_{1,n}]\|_{L^{2\frac{d+1}{d-1}}} \underset{n\to\infty}{\longrightarrow} W(d)$. By the profile decomposition (2.7) together with (2.8),

$$W(d)^{2\frac{d+1}{d-1}} = \limsup_{n\to\infty} \|S[u_{0,n}, u_{1,n}]\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = \lim_{N\to\infty} \limsup_{n\to\infty} \|\sum_{j=1}^{N} \Gamma_j^n S[\phi_0^j, \phi_1^j]\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.$$

By Lemma 2.1, this is equal to

$$\sum_{j=1}^{\infty} \|S[\phi_0^j, \phi_1^j]\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.$$

Using the Strichartz inequality (2.2) and (2.9), this is bounded by

$$W(d)^{2\frac{d+1}{d-1}} \sum_{j=1}^{\infty} \|(\phi_0^j, \phi_1^j)\|^{2\frac{d+1}{d-1}}_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} \le W(d)^{2\frac{d+1}{d-1}} (\sum_{j=1}^{\infty} \|(\phi_0^j, \phi_1^j)\|^{2}_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}})^{\frac{d+1}{d-1}} \le W(d)^{2\frac{d+1}{d-1}}.$$

Therefore, in order to have equalities throughout, there should be exactly one term in the sum, which yields the maximizing pair.

$\square$

The proof of Theorem 2.1 will follow from the following proposition concerning compactness. We first define the inverse transformation $(\Gamma_j^n)^{-1}$ associated to $(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)$ as:

$$(\Gamma_j^n)^{-1} F(x,t) = \Big(\frac{\ell_j^n}{r_j^n}\Big)^{\frac{d-1}{2}} F\Big(T^{\ell_j^n}_{w_j^n} \frac{1}{r_j^n}(x,t) + (x_j^n, t_j^n)\Big).$$

We observe that $(\Gamma_j^n)^{-1}\Gamma_j^n F = F$.

Proposition 2.1. *Let $d \ge 2$, and let $(u_{0,n}, u_{1,n})_n$ be a sequence in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ such that*

$$\|(u_{0,n}, u_{1,n})\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} \le M \quad \text{and} \quad \|S[u_{0,n}, u_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \ge K.$$

Then, there exists a sequence $(r_{j_0}^n, \ell_{j_0}^n, w_{j_0}^n, x_{j_0}^n, t_{j_0}^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+\backslash\{0\}\times[1,\infty)\times\mathbb{S}^{d-1}\times\mathbb{R}^d\times\mathbb{R}$ such that, up to a subsequence,

$$(\Gamma_{j_0}^n)^{-1} S[u_{0,n}, u_{1,n}] \underset{n\to\infty}{\rightharpoonup} \mathtt{U} \quad \text{with} \quad \|\mathtt{U}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \ge C(K,M).$$

The proof of Proposition 2.1 will occupy the next section.

2.2. Proof of Proposition 2.1

We will require two propositions before starting the proof of Proposition 2.1. The first one gives a statement similar to Theorem 2.1 but under the stronger hypothesis of localized frequency of the sequence. The principle arguments of the proof can be traced back to [57], [33] and [1]. We will need the following lemma, a proof of which can be found in [58] for the Schrödinger equation. The same proof works in this case.

Lemma 2.3. *Let $(\phi_{0,n},\phi_{1,n})_n$ and (ϕ_0,ϕ_1) be in $\dot H^{\frac12}\times\dot H^{-\frac12}$. The following statements are equivalent:*

(i) $(\phi_{0,n},\phi_{1,n}) \underset{n\to\infty}{\rightharpoonup} (\phi_0,\phi_1)$ *weakly in* $\dot H^{\frac12}\times\dot H^{-\frac12}(\mathbb{R}^d)$.

(ii) $S[\phi_{0,n},\phi_{1,n}] \underset{n\to\infty}{\rightharpoonup} S[\phi_0,\phi_1]$ *weakly in* $L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})$.

Proposition 2.2. *Let $d\geq 2$ and $(P_{0,n},P_{1,n})_n$ be a bounded sequence in $\dot H^{\frac12}\times\dot H^{-\frac12}(\mathbb{R}^d)$ such that*

$$|\widehat{P_{0,n}}|, |\widehat{P_{1,n}}| \lesssim \chi_F, \tag{2.10}$$

where $F\subset\mathbb{R}^d\setminus\{0\}$ is a compact set.

Then, there exist a subsequence (still denoted $(P_{0,n},P_{1,n})_n$), a sequence $(\phi_0^\alpha,\phi_1^\alpha)_{\alpha\in\mathbb{N}}$, and pairs $\{(y_\alpha^n,s_\alpha^n)_{n\in\mathbb{N}}\}_{\alpha\in\mathbb{N}}$ in $\mathbb{R}^d\times\mathbb{R}$, obeying

$$|y_\alpha^n - y_{\alpha'}^n| + |s_\alpha^n - s_{\alpha'}^n| \underset{n\to\infty}{\longrightarrow} +\infty, \quad \text{for every } \alpha\neq\alpha' \tag{2.11}$$

such that

$$S[P_{0,n},P_{1,n}](x,t) = \sum_{\alpha=1}^{A} S[\phi_0^\alpha,\phi_1^\alpha](x-y_\alpha^n, t-s_\alpha^n) + S[\mathtt{P}_{0,n}^A,\mathtt{P}_{1,n}^A](x,t), \tag{2.12}$$

with

$$\lim_{n\to\infty} \|S[\mathtt{P}_{0,n}^A,\mathtt{P}_{1,n}^A]\|_{2\frac{d+1}{d-1}(\mathbb{R}^{d+1})} \underset{A\to+\infty}{\longrightarrow} 0, \tag{2.13}$$

$$|\widehat{\phi_0^\alpha}|, |\widehat{\phi_1^\alpha}| \leq \chi_F \quad \text{for every } \alpha, \tag{2.14}$$

and for every $A\geq 1$, the orthogonality property

$$\|(P_{0,n},P_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}} = \sum_{\alpha=1}^{A} \|(\phi_0^\alpha,\phi_1^\alpha)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}} + \|(\mathtt{P}_{0,n}^A,\mathtt{P}_{1,n}^A)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}} + o(1) \text{ as } n\to\infty\,. \tag{2.15}$$

Proof. Letting $(\mathbf{P_0}, \mathbf{P_1}) = (P_{0,n}, P_{1,n})_n$ be a bounded sequence in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$, we define the set $\mathcal{V}(\mathbf{P_0}, \mathbf{P_1})$ by

$$\mathcal{V}(\mathbf{P_0}, \mathbf{P_1}) = \left\{ \begin{array}{c|c} (\phi_0, \phi_1) & \text{there exists a family of pairs } (x_n, t_n) \\ \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} & \text{such that, up to a subsequence :} \\ & S[P_{0,n}, P_{1,n}](x + x_n, t_n) \underset{n\to\infty}{\rightharpoonup} \phi_0 \text{ weakly in } \dot{H}^{\frac{1}{2}} \\ & \partial_t S[P_{0,n}, P_{1,n}](x + x_n, t_n) \underset{n\to\infty}{\rightharpoonup} \phi_1 \text{ weakly in } \dot{H}^{-\frac{1}{2}}. \end{array} \right\},$$

and write

$$\eta(\mathbf{P_0}, \mathbf{P_1}) = \sup \left\{ \|(\phi_0, \phi_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \quad ; \quad (\phi_1, \phi_2) \in \mathcal{V}(\mathbf{P_0}, \mathbf{P_1}) \right\}.$$

As $(\mathbf{P_0}, \mathbf{P_1})$ is bounded, the set $\mathcal{V}(\mathbf{P_0}, \mathbf{P_1})$ is not empty just by taking the sequence $(x_n, t_n) = (0, 0)$.

We begin by proving

$$\limsup_{n\to\infty} \|S[P_{0,n}, P_{1,n}]\|_{2\frac{d+1}{d-1}(\mathbb{R}^{d+1})} \lesssim \eta(\mathbf{P_0}, \mathbf{P_1})^{\theta} \quad \text{for some } \theta > 0. \tag{2.16}$$

Using Wolff's linear restriction Theorem [84], we have for some $p_0 < 2$, $q_0 < 2\frac{d+1}{d-1}$,

$$\begin{aligned} \limsup_{n\to\infty} \|S[P_{0,n}, P_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} &\leq \limsup_{n\to\infty} \|S[P_{0,n}, P_{1,n}]\|_{L^{q_0}(\mathbb{R}^{d+1})}^{\frac{q_0(d-1)}{2(d+1)}} \|S[P_{0,n}, P_{1,n}]\|_{L^{\infty}(\mathbb{R}^{d+1})}^{1-\frac{q_0(d-1)}{2(d+1)}} \\ &\lesssim \limsup_{n\to\infty} (\|\widehat{P_{0,n}}\|_{L^{p_0}(\mathbb{R}^d)} + \|\widehat{P_{1,n}}\|_{L^{p_0}(\mathbb{R}^d)})^{\frac{q_0(d-1)}{2(d+1)}} \|S[P_{0,n}, P_{1,n}]\|_{L^{\infty}(\mathbb{R}^{d+1})}^{1-\frac{q_0(d-1)}{2(d+1)}}. \end{aligned}$$

Using (2.10), this yields

$$\limsup_{n\to\infty} \|S[P_{0,n}, P_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \lesssim \limsup_{n\to\infty} \|S[P_{0,n}, P_{1,n}]\|_{L^{\infty}(\mathbb{R}^{d+1})}^{1-\frac{q_0(d-1)}{2(d+1)}}. \tag{2.17}$$

Now, by the compact Fourier support of $P_{0,n}, P_{1,n}$, and Remark 1.1, we can deduce that for some $\widehat{\psi} \in C_0^{\infty}(\mathbb{R}^{d+1})$, we have

$$S[P_{0,n}, P_{1,n}] = S[P_{0,n}, P_{1,n}] * \psi.$$

Hence, there exist (x_n, t_n) such that

$$\begin{aligned} &\limsup_{n\to\infty} \|S[P_{0,n}, P_{1,n}]\|_{L^{\infty}(\mathbb{R}^{d+1})} \lesssim \limsup_{n\to\infty} |(S[P_{0,n}, P_{1,n}] * \psi)(x_n, t_n)| \\ &= \limsup_{n\to\infty} \left| \iint \psi(-x, -t) S[P_{0,n}, P_{1,n}](x + x_n, t + t_n) dxdt \right| \Big\} \\ &= \limsup_{n\to\infty} \left| \iint \psi(-x, -t) S\Big[S[P_{0,n}, P_{1,n}](\cdot + x_n, t_n), \partial_t S[P_{0,n}, P_{1,n}](\cdot + x_n, t_n)\Big](x, t) dxdt \right|. \end{aligned}$$

Using Lemma 2.3 this is bounded by

$$\sup\Big\{\Big|\int\int\psi(-x,-t)S[\phi_0,\phi_1](x,t)dxdt\Big| : (\phi_0,\phi_1)\in\mathcal{V}(\mathbf{P_0},\mathbf{P_1})\Big\}.$$

By Hölder's inequality and the Strichartz inequality (2.2), this is bounded by a constant multiple of

$$\begin{aligned}&\sup\Big\{\|(\phi_0,\phi_1)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}} : (\phi_0,\phi_1)\in\mathcal{V}(\mathbf{P_0},\mathbf{P_1})\Big\}\\ &\lesssim \eta(\mathbf{P_0},\mathbf{P_1})^2,\end{aligned}$$

which yields (2.16).

We extract now the functions $\phi_0^\alpha,\phi_1^\alpha$ recursively. If $\eta(\mathbf{P_0},\mathbf{P_1})=0$, then by (2.16) we can take $\phi_0^\alpha\equiv 0$, $\phi_1^\alpha\equiv 0$ for all α and we are done. Otherwise, there exist $(\phi_0^1,\phi_1^1)\in\mathcal{V}(\mathbf{P_0},\mathbf{P_1})$ such that

$$\|(\phi_0^1,\phi_1^1)\|_{\dot H^{\frac12}\times\dot H^{-\frac12}}\ge\frac12\eta(\mathbf{P_0},\mathbf{P_1})>0.$$

By the definition, we can choose a sequence $(y_1^n,s_1^n)\subset\mathbb{R}^d\times\mathbb{R}$ such that, up to extracting a subsequence, we have:

$$\begin{aligned}S[P_{0,n},P_{1,n}](x+y_1^n,s_1^n)&\underset{n\to\infty}{\rightharpoonup}\phi_0^1\quad\text{weakly in }\dot H^{\frac12},\\ \partial_tS[P_{0,n},P_{1,n}](x+y_1^n,s_1^n)&\underset{n\to\infty}{\rightharpoonup}\phi_1^1\quad\text{weakly in }\dot H^{-\frac12},\end{aligned}$$

where we observe that the functions ϕ_0^1,ϕ_1^1 have Fourier support contained in F. We set

$$\begin{aligned}\mathtt{P}^1_{0,n}(x)&:=P_{0,n}(x)-S[\phi_0^1,\phi_1^1](x-y_1^n,-s_1^n),\\ \mathtt{P}^1_{1,n}(x)&:=P_{1,n}(x)-\partial_tS[\phi_0^1,\phi_1^1](x-y_1^n,-s_1^n),\end{aligned}$$

so that

$$S[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](x+y_1^n,s_1^n)\underset{n\to\infty}{\rightharpoonup}0\quad\text{and}\quad\partial_tS[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](x+y_1^n,s_1^n)\underset{n\to\infty}{\rightharpoonup}0,\tag{2.18}$$

and that $\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}$ have Fourier support contained in F. Now, for ψ with compact Fourier support we have

$$\begin{aligned}&\|\psi*P_{0,n}\|^2_{\dot H^{\frac12}}+\|\psi*P_{1,n}\|^2_{\dot H^{-\frac12}}\\ &=\big\|S[\psi*\phi_0^1,\psi*\phi_1^1](\cdot-y_1^n,-s_1^n)\big\|^2_{\dot H^{\frac12}}+\big\|\partial_tS[\psi*\phi_0^1,\psi*\phi_1^1](\cdot-y_1^n,-s_1^n)\big\|^2_{\dot H^{-\frac12}}\\ &+\big\|\psi*\mathtt{P}^1_{0,n}\big\|^2_{\dot H^{\frac12}}+\big\|\psi*\mathtt{P}^1_{1,n}\big\|^2_{\dot H^{-\frac12}}+2\big\langle\psi*\mathtt{P}^1_{0,n},S[\psi*\phi_0^1,\psi*\phi_1^1](\cdot-y_1^n,-s_1^n)\big\rangle_{\dot H^{\frac12}}\\ &+2\big\langle\psi*\mathtt{P}^1_{1,n},\partial_tS[\psi*\phi_0^1,\psi*\phi_1^1](\cdot-y_1^n,-s_1^n)\big\rangle_{\dot H^{-\frac12}}\\ &=\big\|\psi*\phi_0^1\big\|^2_{\dot H^{\frac12}}+\big\|\psi*\phi_1^1\big\|^2_{\dot H^{-\frac12}}+\big\|\psi*\mathtt{P}^1_{0,n}\big\|^2_{\dot H^{\frac12}}+\big\|\psi*\mathtt{P}^1_{1,n}\big\|^2_{\dot H^{-\frac12}}\\ &+2\big\langle\psi*\phi_0^1,S[\psi*\mathtt{P}^1_{0,n},\psi*\mathtt{P}^1_{1,n}](\cdot+y_1^n,s_1^n)\big\rangle_{\dot H^{\frac12}}\\ &+2\big\langle\psi*\phi_1^1,\partial_tS[\psi*\mathtt{P}^1_{0,n},\psi*\mathtt{P}^1_{1,n}](\cdot+y_1^n,s_1^n)\big\rangle_{\dot H^{-\frac12}}.\end{aligned}$$

Hence, using (2.18), we have

$$\begin{aligned}&\big\|(\psi*P_{0,n},\psi*P_{1,n})\big\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}\\&\quad=\big\|(\psi*\phi_0^1,\psi*\phi_1^1)\big\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}+\big\|(\psi*\mathtt{P}^1_{0,n},\psi*\mathtt{P}^1_{1,n})\big\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}+o(1)\quad\text{as } n\to\infty.\end{aligned}$$

Therefore, taking ψ appropriately and by (2.10), we conclude that $|\widehat{\phi_0^1}(\xi)|$, $|\widehat{\phi_1^1}(\xi)|$, $|\widehat{\mathtt{P}^1_{0,n}}(\xi)|$, $|\widehat{\mathtt{P}^1_{1,n}}(\xi)|\lesssim 1$ almost everywhere.

If we take $\widehat{\psi}\equiv 1$ in the set F,

$$\big\|(P_{0,n},P_{1,n})\big\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}=\big\|(\phi_0^1,\phi_1^1)\big\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}+\big\|(\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n})\big\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}+o(1).$$

Now, we repeat the above process replacing $(P_{0,n},P_{1,n})_n$ with $(\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n})_n$, observing that the hypothesis on $(P_{0,n},P_{1,n})_n$ are also satisfied by $(\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n})_n$. If $\eta(\mathtt{P}^{\mathbf{1}}_{\mathbf{0}},\mathtt{P}^{\mathbf{1}}_{\mathbf{1}})>0$, we get ϕ_0^2,ϕ_1^2, (y_2^n,s_2^n) and $(\mathtt{P}^2_{0,n},\mathtt{P}^2_{1,n})_n$.

To see that $|s_1^n-s_2^n|+|y_1^n-y_2^n|\underset{n\to\infty}{\longrightarrow}\infty$ we suppose otherwise. We could then find a subsequence (still indexed by n) such that

$$s_1^n-s_2^n=s_*^n\quad s_*^n\to s_*.\quad\text{and}\quad y_1^n-y_2^n=y_*^n,\quad y_*^n\to y_*.$$

So that for every pair $(h_1,h_2)\in\dot H^{\frac12}\times\dot H^{-\frac12}$,

$$\begin{aligned}&\langle S[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_2^n,s_2^n),h_1\rangle_{\dot H^{\frac12}}+\langle\partial_tS[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_2^n,s_2^n),h_2\rangle_{\dot H^{-\frac12}}\\&\quad=\langle S[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_1^n,s_1^n),S[h_1,h_2](\cdot+y_*^n,s_*^n)\rangle_{\dot H^{\frac12}}\\&\qquad+\langle\partial_tS[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_1^n,s_1^n),\partial_tS[h_1,h_2](\cdot+y_*^n,s_*^n)\rangle_{\dot H^{-\frac12}}.\end{aligned}$$

Thus by (2.18) and the strong convergence of $S[h_1,h_2](\cdot+y_*^n,s_*^n)\to S[h_1,h_2](\cdot+y_*,s_*)$ and $\partial_tS[h_1,h_2](\cdot+y_*^n,s_*^n)\to\partial_tS[h_1,h_2](\cdot+y_*,s_*)$, we get

$$\langle S[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_2^n,s_2^n),h_1\rangle_{\dot H^{\frac12}}+\langle\partial_tS[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_2^n,s_2^n),h_2\rangle_{\dot H^{-\frac12}}\to 0.$$

Recalling that $S[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_2^n,s_2^n)\rightharpoonup\phi_0^2$, $\partial_tS[\mathtt{P}^1_{0,n},\mathtt{P}^1_{1,n}](\cdot+y_2^n,s_2^n)\rightharpoonup\phi_1^2$, the uniqueness of weak limits would imply that $\phi_0^2=0$ and $\phi_1^2=0$, and therefore $\eta(\mathtt{P}^{\mathbf{1}}_{\mathbf{0}},\mathtt{P}^{\mathbf{1}}_{\mathbf{1}})=0$, which gives a contradiction. Iterating the process we get the pairs $(\phi_0^\alpha,\phi_1^\alpha)_\alpha$, $(y_\alpha^n,s_\alpha^n)_\alpha$ satisfying (2.11), (2.12), (2.14) and (2.15). It remains to prove (2.13). Since $(P_{0,n},P_{1,n})_n$ is bounded in $\dot H^{\frac12}\times\dot H^{-\frac12}$ and by (2.15),

$$\sum_\alpha\|(\phi_0^\alpha,\phi_1^\alpha)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}\le\limsup_{n\to\infty}\|(P_{0,n},P_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}},$$

the series $\displaystyle\sum_\alpha\|(\phi_0^\alpha,\phi_1^\alpha)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}$ converges, so that

$$\|(\phi_0^\alpha,\phi_1^\alpha)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}\underset{\alpha\to\infty}{\longrightarrow}0.$$

Now, by construction we have

$$\|(\phi_0^\alpha, \phi_1^\alpha)\|^2_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \geq \frac{1}{2}\eta(\mathtt{P}_{\mathbf{0}}^{\alpha-\mathbf{1}}, \mathtt{P}_{\mathbf{1}}^{\alpha-\mathbf{1}}),$$

so that

$$\eta(\mathtt{P}_{\mathbf{0}}^{\mathbf{A}}, \mathtt{P}_{\mathbf{1}}^{\mathbf{A}}) \underset{A\to\infty}{\longrightarrow} 0,$$

and we are done by (2.16).

$\square$

Now, we extract the cores of our sequences, enabling us to satisfy the hypothesis of Proposition 2.2. The key ingredient will be the Strichartz refinement proved in the first chapter. The proof of the following proposition is an adaptation of a result in [**11**] (see also [**58**]).

Proposition 2.3. *Let $(u_{0,n}, u_{1,n})_n$ be a bounded sequence in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ with $d \geq 2$. Then, for every $\epsilon > 0$, there exist $N = N(\epsilon, \|\mathbf{u_0}, \mathbf{u_1}\|)$, a family $\{(g_{0,n}^i, g_{1,n}^i)\}_{1\leq i\leq N}$ and a family of sequences $\{(2^{k_i^n}, 2^{j_i^n}, \theta_i^n)_{n\in\mathbb{N}}\}_{1\leq i\leq N}$ in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1}$ that satisfy, up to a subsequence,*

(i) *The rescaling, Lorentz or angular property:*

$$\frac{2^{k_i^n}}{2^{k_{i'}^n}} + \frac{2^{k_{i'}^n}}{2^{k_i^n}} + \frac{2^{j_i^n}}{2^{j_{i'}^n}} + \frac{2^{j_{i'}^n}}{2^{j_i^n}} + 2^{j_i^n}|\theta_i^n - \theta_{i'}^n| \underset{n\to\infty}{\to} \infty \qquad \forall\, i \neq i'.$$

(ii) *Compact Fourier support:*

$$\mathrm{supp}(\widehat{g}_{0,n}^i),\ \mathrm{supp}(\widehat{g}_{1,n}^i)\ \subset \mathcal{T}_i^n,$$

with

$$\big\{(\xi, |\xi|) \in \mathbb{R}^{d+1} : \quad (\xi, |\xi|) = T_{\theta_i^n}^{2^{j_i^n}} \frac{1}{2^{k_i^n}}(\rho, |\rho|), \quad \rho \in \mathcal{T}_i^n\big\}$$

contained in a compact set, independent of n and i, that does not contain the origin.

(iii) *Boundedness: there exists a $C = C(\epsilon, (\mathbf{u_0}, \mathbf{u_1}))$ such that*

$$2^{\frac{k_i^n}{2}}|\widehat{g}_{0,n}^i|, 2^{\frac{-k_i^n}{2}}|\widehat{g}_{1,n}^i| \leq C|\mathcal{T}_i^n|^{-\frac{1}{2}}.$$

(iv) *The smallness property:*

$$\|S[u_{0,n}, u_{1,n}] - \sum_i^N S[g_{0,n}^i, g_{1,n}^i]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \epsilon.$$

(v) *The almost orthogonality identity:*

$$\begin{aligned}\|(u_{0,n},u_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}} &= \sum_{i=1}^{N}\|(g^i_{0,n},g^i_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}\\ &\quad+\|(u_{0,n}-\sum_{i=1}^{N}g^i_{0,n},u_{1,n}-\sum_{i=1}^{N}g^i_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}.\end{aligned}$$

Proof. Suppose first that $\|S[u_{0,n},u_{1,n}]\|_{L^{2\frac{d+1}{d-1}}}\geq\epsilon$. By Theorem 1.1 and the expression (2.1) we deduce that there exist $p<2$ and $0<\theta<1$, for which

$$\begin{aligned}\|S[u_{0,n},u_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} &\lesssim \sup_{k,j,m}2^{k\frac{\theta}{2}}|\tau^{j,k}_m|^{\frac{\theta}{2}\frac{p-2}{p}}(\int_{\tau^{j,k}_m}|\widehat{P_k u_{0,n}}|^p)^{\frac{\theta}{p}}\|u_{0,n}\|^{1-\theta}_{\dot H^{\frac12}}\\ &+\quad\sup_{k,j,m}2^{-k\frac{\theta}{2}}|\tau^{j,k}_m|^{\frac{\theta}{2}\frac{p-2}{p}}(\int_{\tau^{j,k}_m}|\widehat{P_k u_{1,n}}|^p)^{\frac{\theta}{p}}\|u_{1,n}\|^{1-\theta}_{\dot H^{-\frac12}}\\ &\lesssim 2^{k_1^n\frac{\theta}{2}}|\tau^{j_1^n,k_1^n}_{m_1^n}|^{\frac{\theta}{2}\frac{p-2}{p}}(\int_{\tau^{j_1^n,k_1^n}_{m_1^n}}|\widehat{P_{k_1^n}u_{0,n}}|^p)^{\frac{\theta}{p}}\|u_{0,n}\|^{1-\theta}_{\dot H^{\frac12}}\\ &+\;2^{-k_1^n\frac{\theta}{2}}|\tau^{j_1^n,k_1^n}_{m_1^n}|^{\frac{\theta}{2}\frac{p-2}{p}}(\int_{\tau^{j_1^n,k_1^n}_{m_1^n}}|\widehat{P_{k_1^n}u_{1,n}}|^p)^{\frac{\theta}{p}}\|u_{1,n}\|^{1-\theta}_{\dot H^{-\frac12}},\end{aligned}$$

for some (k_1^n,j_1^n,m_1^n). Here we used the fact that $|a|+|b|\leq 2\max\{|a|,|b|\}$.

Setting $u'_{0,n}=2^{\frac{k_1^n}{2}}u_{0,n}$, $u'_{1,n}=2^{-\frac{k_1^n}{2}}u_{1,n}$, we have

$$\begin{aligned}\epsilon &\lesssim |\tau^{j_1^n,k_1^n}_{m_1^n}|^{\frac{\theta}{2}\frac{p-2}{p}}(\int_{\tau^{j_1^n,k_1^n}_{m_1^n}}|\widehat{P_{k_1^n}u'_{0,n}}|^p)^{\frac{\theta}{p}}\|u_{0,n}\|^{1-\theta}_{\dot H^{\frac12}}\\ &+|\tau^{j_1^n,k_1^n}_{m_1^n}|^{\frac{\theta}{2}\frac{p-2}{p}}(\int_{\tau^{j_1^n,k_1^n}_{m_1^n}}|\widehat{P_{k_1^n}u'_{1,n}}|^p)^{\frac{\theta}{p}}\|u_{1,n}\|^{1-\theta}_{\dot H^{-\frac12}}.\end{aligned}$$

Thus,

$$\int_{\tau^{j_1^n,k_1^n}_{m_1^n}}|\widehat{P_{k_1^n}u'_{0,n}}|^p+|\widehat{P_{k_1^n}u'_{1,n}}|^p\gtrsim\epsilon^{\frac{p}{\theta}}\Big[\max(\|u_{0,n}\|_{\dot H^{\frac12}},\|u_{1,n}\|_{\dot H^{-\frac12}})\Big]^{p-\frac{p}{\theta}}|\tau^{j_1^n,k_1^n}_{m_1^n}|^{1-\frac{p}{2}}. \tag{2.19}$$

We observe that $p-\frac{p}{\theta}<0$, and define

$$c_n=\epsilon^{\frac{p}{\theta}}\Big[\max(\|u_{0,n}\|_{\dot H^{\frac12}},\|u_{1,n}\|_{\dot H^{-\frac12}})\Big]^{p-\frac{p}{\theta}}. \tag{2.20}$$

On the other hand,

$$\int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|>\lambda\right\}}|\widehat{P_{k_1^n}u'_{0,n}}|^p \leq \int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|>\lambda\right\}}|\widehat{P_{k_1^n}u'_{0,n}}|^p\left(\frac{|\widehat{P_{k_1^n}u'_{0,n}}|}{\lambda}\right)^{2-p}$$
$$= \frac{\|P_{k_1^n}u'_{0,n}\|_2^2}{\lambda^{2-p}} \leq \frac{\|u_{0,n}\|^2_{\dot H^{\frac12}}}{\lambda^{2-p}},$$

and similarly

$$\int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{1,n}}|>\lambda\right\}}|\widehat{P_{k_1^n}u'_{1,n}}|^p \leq \frac{\|u_{1,n}\|^2_{\dot H^{-\frac12}}}{\lambda^{2-p}}.$$

Therefore, setting

$$\lambda = \left(\frac{4\max(\|u_{0,n}\|^2_{\dot H^{\frac12}}, \|u_{1,n}\|^2_{\dot H^{-\frac12}})}{c_n}\right)^{\frac{1}{2-p}}|\tau_{m_1^n}^{j_1^n,k_1^n}|^{-\frac12},$$

we have by (2.19), that

$$\int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|\leq\lambda\right\}}|\widehat{P_{k_1^n}u'_{0,n}}|^p + \int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{1,n}}|\leq\lambda\right\}}|\widehat{P_{k_1^n}u'_{1,n}}|^p$$
$$= \int_{\tau_{m_1^n}^{j_1^n,k_1^n}}|\widehat{P_{k_1^n}u'_{0,n}}|^p + |\widehat{P_{k_1^n}u'_{1,n}}|^p$$
$$- \Big(\int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|>\lambda\right\}}|\widehat{P_{k_1^n}u'_{0,n}}|^p + \int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{1,n}}|>\lambda\right\}}|\widehat{P_{k_1^n}u'_{1,n}}|^p\Big) \geq \frac{c_n}{2}|\tau_{m_1^n}^{j_1^n,k_1^n}|^{1-\frac p2},$$

By Hölder's inequality

(2.21)
$$\left(\int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|\leq\lambda\right\}}|\widehat{P_{k_1^n}u'_{0,n}}|^2 + \int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{1,n}}|\leq\lambda\right\}}|\widehat{P_{k_1^n}u'_{1,n}}|^2\right)^{\frac p2}$$
$$\geq |\tau_{m_1^n}^{j_1^n,k_1^n}|^{\frac p2-1}\Big(\int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|\leq\lambda\right\}}|\widehat{P_{k_1^n}u'_{0,n}}|^p + \int_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{1,n}}|\leq\lambda\right\}}|\widehat{P_{k_1^n}u'_{1,n}}|^p\Big) \geq \frac{c_n}{2}.$$

Now, defining

$$\widehat{f}^1_{0,n} = 2^{-\frac{k_1^n}{2}}\widehat{P_{k_1^n}u'_{0,n}}\chi_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{0,n}}|\leq\lambda\right\}} = \widehat{P_{k_1^n}u_{0,n}}\chi_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u_{0,n}}|\leq 2^{-\frac{k_1^n}{2}}\lambda\right\}},$$
$$\widehat{f}^1_{1,n} = 2^{\frac{k_1^n}{2}}\widehat{P_{k_1^n}u'_{1,n}}\chi_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u'_{1,n}}|\leq\lambda\right\}} = \widehat{P_{k_1^n}u_{1,n}}\chi_{\tau_{m_1^n}^{j_1^n,k_1^n}\cap\left\{|\widehat{P_{k_1^n}u_{1,n}}|\leq 2^{\frac{k_1^n}{2}}\lambda\right\}},$$

these functions are supported in a set $\tau_{m_1^n}^{j_1^n,k_1^n}$, and

$$|\widehat{f}_{0,n}^1| \leq 2^{-\frac{k_1^n}{2}}\lambda = 2^{-\frac{k_1^n}{2}}A|\tau_{m_1^n}^{j_1^n,k_1^n}|^{-\frac{1}{2}}, \quad |\widehat{f}_{1,n}^1| \leq 2^{\frac{k_1^n}{2}}\lambda = 2^{\frac{k_1^n}{2}}A|\tau_{m_1^n}^{j_1^n,k_1^n}|^{-\frac{1}{2}},$$

where

$$A = \epsilon^{-\frac{p}{\theta(2-p)}}\max(\|u_{0,n}\|_{\dot{H}^{\frac{1}{2}}}, \|u_{1,n}\|_{\dot{H}^{-\frac{1}{2}}})^{1+\frac{p}{\theta(2-p)}}.$$

Moreover,

$$2^{k_1^n}\int |\widehat{f}_{0,n}^1|^2 + 2^{-k_1^n}\int |\widehat{f}_{1,n}^1|^2 \geq \left(\frac{C_n}{2}\right)^{\frac{2}{p}}.$$

We define now

$$(f_{0,n,1}, f_{1,n,1}) = (u_{0,n}, u_{1,n}) - (f_{0,n}^1, f_{1,n}^1).$$

If

$$\|S[u_{0,n}, u_{1,n}] - S[f_{0,n}^1, f_{1,n}^1]\|_{L^{2\frac{d+1}{d-1}}} = \|S[f_{0,n,1}, f_{1,n,1}]\|_{L^{2\frac{d+1}{d-1}}} < \epsilon,$$

we are done. If not, we repeat the process with $(f_{0,n,1}, f_{1,n,1})$. And recursively we obtain functions $(f_{0,n,i}, f_{1,n,i}) = (f_{0,n,i-1}, f_{1,n,i-1}) - (f_{0,n}^i, f_{1,n}^i)$. We observe that the $(\widehat{f}_{0,n}^i, \widehat{f}_{1,n}^i)_i$ have disjoint supports. The functions $\widehat{f}_{0,n}^i$, $\widehat{f}_{1,n}^i$ are compactly supported on some sets $\tau_{m_i^n}^{j_i^n,k_i^n}$. This is similar to (ii) with $g_{0,n}^i$, $g_{1,n}^i$ replaced by $f_{0,n}^i$, $f_{1,n}^i$, and $\mathcal{T}_i^n$ replaced by $\tau_{m_i^n}^{j_i^n,k_i^n}$.

As $\|f_{0,n,i}\|_{\dot{H}^{\frac{1}{2}}} \leq \|u_{0,n}\|_{\dot{H}^{\frac{1}{2}}}$, $\|f_{1,n,i}\|_{\dot{H}^{\frac{1}{2}}} \leq \|u_{1,n}\|_{\dot{H}^{-\frac{1}{2}}}$, we see that

$$\begin{aligned} |\widehat{f}_{0,n}^i| &\leq 2^{-\frac{k_i^n}{2}}\epsilon^{-\frac{p}{\theta(2-p)}}\max(\|f_{0,n,i-1}\|_{\dot{H}^{\frac{1}{2}}}, \|f_{1,n,i-1}\|_{\dot{H}^{-\frac{1}{2}}})^{1+\frac{p}{\theta(2-p)}}|\tau_{m_i^n}^{j_i^n,k_i^n}|^{-\frac{1}{2}} \\ &\leq 2^{-\frac{k_i^n}{2}}A|\tau_{m_i^n}^{j_i^n,k_i^n}|^{-\frac{1}{2}} \\ |\widehat{f}_{1,n}^i| &\leq 2^{\frac{k_i^n}{2}}\epsilon^{-\frac{p}{\theta(2-p)}}\max(\|f_{0,n,i-1}\|_{\dot{H}^{\frac{1}{2}}}, \|f_{1,n,i-1}\|_{\dot{H}^{-\frac{1}{2}}})^{1+\frac{p}{\theta(2-p)}}|\tau_{m_i^n}^{j_i^n,k_i^n}|^{-\frac{1}{2}} \\ &\leq 2^{\frac{k_i^n}{2}}A|\tau_{m_i^n}^{j_i^n,k_i^n}|^{-\frac{1}{2}}. \end{aligned} \tag{2.22}$$

This corresponds to (iii). Recalling that $p - \frac{p}{\theta} < 0$, we also have that

(2.23)
$$2^{k_i^n}\int |\widehat{f}_{0,n}^i|^2 + 2^{-k_i^n}\int |\widehat{f}_{1,n}^i|^2 \geq (\epsilon^{\frac{p}{\theta}}\max(\|f_{0,n,i-1}\|_{\dot{H}^{\frac{1}{2}}}, \|f_{1,n,i-1}\|_{\dot{H}^{-\frac{1}{2}}})^{p-\frac{p}{\theta}})^{\frac{2}{p}} \geq \left(\frac{C_n}{2}\right)^{\frac{2}{p}}.$$

Observe that $\widehat{u_{0,n}} - \sum_i^{N_n} \widehat{f}^i_{0,n}$ and $\sum_i^{N_n} \widehat{f}^i_{0,n}$ have disjoints supports, as well as $\widehat{u_{1,n}} - \sum_i^{N_n} \widehat{f}^i_{1,n}$ and $\sum_i^{N_n} \widehat{f}^i_{1,n}$. Thus

$$\|\widehat{u_{0,n}} - \sum_i^{N_n} \widehat{f}^i_{0,n}\|_2^2 = \|\widehat{u_{0,n}}\|_2^2 - \|\sum_i^{N_n} \widehat{f}^i_{0,n}\|_2^2 = \|\widehat{u_{0,n}}\|_2^2 - \sum_i^{N_n} \|\widehat{f}^i_{0,n}\|_2^2, \tag{2.24}$$

$$\|\widehat{u_{1,n}} - \sum_i^{N_n} \widehat{f}^i_{1,n}\|_2^2 = \|\widehat{u_{1,n}}\|_2^2 - \|\sum_i^{N_n} \widehat{f}^i_{1,n}\|_2^2 = \|\widehat{u_{1,n}}\|_2^2 - \sum_i^{N_n} \|\widehat{f}^i_{1,n}\|_2^2. \tag{2.25}$$

Finally, using the Strichartz inquality (2.2),

$$\begin{aligned}
\|S[u_{0,n}, u_{1,n}] - \sum_i^{N_n} S[f^i_{0,n}, f^i_{1,n}]\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \\
&= \|S[u_{0,n} - \sum_i^{N_n} f^i_{0,n}, u_{1,n} - \sum_i^{N_n} f^i_{1,n}]\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \\
&\lesssim \sum_k 2^k \|P_k u_{0,n} - \sum_i^{N_n} P_k f^i_{0,n}\|_2^2 + \sum_k 2^{-k} \|P_k u_{1,n} - \sum_i^{N_n} P_k f^i_{1,n}\|_2^2.
\end{aligned}$$

By (2.24), (2.25) and Plancherel's theorem, this is equal to

$$\|u_{0,n}\|^2_{\dot{H}^{\frac{1}{2}}} - \sum_k 2^k \sum_i^{N_n} \|P_k f^i_{0,n}\|_2^2 + \|u_{1,n}\|_{\dot{H}^{-\frac{1}{2}}} - \sum_k 2^{-k} \sum_i^{N_n} \|P_k f^i_{1,n}\|_2^2.$$

As every pair $\widehat{f}^i_{0,n}, \widehat{f}^i_{1,n}$ is supported in an annulus $A_{k_i^n}$, this is equal to

$$\|u_{0,n}\|^2_{\dot{H}^{\frac{1}{2}}} - \sum_i^{N_n} 2^{k_i^n} \|f^i_{0,n}\|_2^2 + \|u_{1,n}\|^2_{\dot{H}^{-\frac{1}{2}}} - \sum_i^{N_n} 2^{-k_i^n} \|f^i_{1,n}\|_2^2.$$

Finally, by (2.23), this is bounded by

$$\lesssim \|u_{0,n}\|^2_{\dot{H}^{\frac{1}{2}}} + \|u_{1,n}\|^2_{\dot{H}^{-\frac{1}{2}}} - N_n(\frac{c_n}{2})^{\frac{2}{p}}.$$

Thus, taking N_n sufficiently large, we conclude (iv) and by (2.24), (2.25) we also conclude (v), replacing $g^i_{0,n}, g^i_{1,n}$ by $f^i_{0,n}, f^i_{1,n}$, and $\mathcal{T}_i^n$ by $\tau^{j_i^n, k_i^n}_{m_i^n}$.

We remark that as $(u_{0,n}, u_{1,n})_n$ is bounded in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$, the sequence c_n defined in (2.20) is bounded below and so the sequence N_n is bounded above. Letting $N = \sup_n N_n$, we set $(f^i_{0,n}, f^i_{1,n}) = (0,0)$ and $(2^{k_i^n}, 2^{j_i^n}, w^n_{m_i}) = (1,1,\mathbf{w})$ with any $\mathbf{w} \in \mathbf{S}^{d-1}$, for $N_n < i \leq N$. Similarly when $\|S[u_{0,n}, u_{1,n}]\|_{L^{2\frac{d+1}{d-1}}} < \epsilon$ we take $(f^i_{0,n}, f^i_{1,n}) = (0,0)$ and $(2^{k_i^n}, 2^{j_i^n}, w^n_{m_i}) = (1,1,\mathbf{w})$ for $1 \leq i \leq N$.

The family $(2^{k_i^n}, 2^{j_i^n}, w_{m_i^n})_{1\leq i\leq N}$ obtained, does not necessarily satisfy (i), but as we will see, it will be enough to reorganize it.

By taking a subsequence, if necessary, we can assume that for every $\ell \neq \ell'$, either

$$\frac{2^{k^n_{\ell'}}}{2^{k^n_\ell}} + \frac{2^{k^n_\ell}}{2^{k^n_{\ell'}}} + \frac{2^{j^n_{\ell'}}}{2^{j^n_\ell}} + \frac{2^{j^n_\ell}}{2^{j^n_{\ell'}}} + 2^{j^n_\ell}|w_{m^n_\ell} - w_{m^n_{\ell'}}| \underset{n\to\infty}{\to} \infty,$$

or

$$\frac{2^{k^n_{\ell'}}}{2^{k^n_\ell}} + \frac{2^{k^n_\ell}}{2^{k^n_{\ell'}}} + \frac{2^{j^n_{\ell'}}}{2^{j^n_\ell}} + \frac{2^{j^n_\ell}}{2^{j^n_{\ell'}}} + 2^{j^n_\ell}|w_{m^n_\ell} - w_{m^n_{\ell'}}| \leq C.$$

We introduce the following equivalence relation: $\ell \sim \ell'$ if

$$\frac{2^{k^n_{\ell'}}}{2^{k^n_\ell}} + \frac{2^{k^n_\ell}}{2^{k^n_{\ell'}}} + \frac{2^{j^n_{\ell'}}}{2^{j^n_\ell}} + \frac{2^{j^n_\ell}}{2^{j^n_{\ell'}}} + 2^{j^n_\ell}|w_{m^n_\ell} - w_{m^n_{\ell'}}| \underset{n\to\infty}{\not\to} \infty,$$

for $0 \leq \ell, \ell' \leq N$. Denoting the equivalence classes by $\{L_i\}_{1\leq i\leq N_L}$, where $N_L \leq N$,

$$g^i_{0,n} = \sum_{\ell\in L_i} f^\ell_{0,n},\ g^i_{1,n} = \sum_{\ell\in L_i} f^\ell_{1,n},\ \text{ and rename } (k^n_i, j^n_i, \theta^n_i) = (k^n_\ell, j^n_\ell, w^n_{m_\ell}) \text{ for some } \ell \in L_i.$$

As $(g^i_{0,n}, g^i_{1,n})_i$ clearly satisfy the properties (iv) and (v), we just need to check the properties (ii) and (iii).

Setting

$$\begin{aligned}
\mathcal{T}^n_i &= \bigcup_{\ell\in L_i} \tau^{j^n_\ell, k^n_\ell}_{m^n_\ell}\\
C_{1,i} = C_{1,i}(\epsilon, (\mathbf{u_0}, \mathbf{u_1})) &= \max_n \max_{\ell,\ell'\in L_i} \left\{\frac{2^{k^n_{\ell'}}}{2^{k^n_\ell}} + \frac{2^{k^n_\ell}}{2^{k^n_{\ell'}}}\right\} < \infty\\
C_{2,i} = C_{2,i}(\epsilon, (\mathbf{u_0}, \mathbf{u_1})) &= \max_n \max_{\ell,\ell'\in L_i} \left\{\frac{2^{j^n_{\ell'}}}{2^{j^n_\ell}} + \frac{2^{j^n_\ell}}{2^{j^n_{\ell'}}}\right\} < \infty\\
C_{3,i} = C_{3,i}(\epsilon, (\mathbf{u_0}, \mathbf{u_1})) &= \max_n \max_{\ell,\ell'\in L_i} \left\{2^{j^n_\ell}|w_{m^n_\ell} - w_{m^n_{\ell'}}|\right\} < \infty,
\end{aligned}$$

the supports of $\widehat{g}^i_{0,n}, \widehat{g}^i_{1,n}$ are contained in $\mathcal{T}^n_i$, and for $\xi \in \mathcal{T}^n_i$ we have $\frac{1}{2^{k^n_i}}(\xi, |\xi|) \subset \bigcup_{\ell\in L_i} \tilde{\tau}^{j^n_\ell, k^n_\ell - k^n_i}_{m^n_\ell}$, which is contained in a compact set supported away from the origin.

Also, for $\xi \in \tau_{m_\ell^n}^{j_\ell^n, k_\ell^n - k_i^n}$ with $\ell \in L_i$, we have

$$\begin{aligned}\angle(\xi, \theta_i^n) &\leq |\theta_i^n - w_{m_\ell^n}^n| + 2^{-j_\ell^n} \\ &\leq 2^{-j_i^n} C_{3,i} + 2^{-j_i^n} C_{2,i} = 2^{-j_i^n}(C_{2,i} + C_{3,i}),\end{aligned}$$

and therefore for $\xi \in \mathcal{T}_i^n$ we have that $T_{\theta_i^n}^{2^{j_i^n}} \dfrac{1}{2^{k_i^n}}(\xi, |\xi|)$ is contained in a compact set independent of i and n, which does not contain the origin. Thus, we get the property (ii).

The property (iii) is clear as we have

$$2^{\frac{k_i^n}{2}} |\widehat{g}_{0,n}^i|, \;\; 2^{-\frac{k_i^n}{2}} |\widehat{g}_{1,n}^i| \leq C_{1,i}^{\frac{d+1}{2}} C_{2,i}^{\frac{d-1}{2}} \; C(\epsilon, \|\mathbf{u_0}, \mathbf{u_1}\|) |\mathcal{T}_i^n|^{-\frac{1}{2}} \leq C(\epsilon, (\mathbf{u_0}, \mathbf{u_1})) |\mathcal{T}_i^n|^{-\frac{1}{2}}.$$

□

In chapter 3, we will require a slightly different version of the previous lemma which we state now. Notice that if we do not require the orthogonality property (i) of Lemma 2.3, the bound on the functions (2.22) depends only on the parameters ϵ and $\|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}$. Thus, the constant which appears in the boundedness property (ii) in the following, depends only on these parameters.

Lemma 2.4. *Let $(u_0, u_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ with $d \geq 2$ and $\|S[u_0, u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq \epsilon$. Then, for every $\epsilon > 0$, there exist $N = N(\epsilon, \|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}})$, $A = A(\epsilon, \|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}})$, a family of pairs of functions $\{(f_0^i, f_1^i)\}_{1 \leq i \leq N}$ and a family of sectors $\{\tau_{m_i}^{j_i, k_i}\}_{1 \leq i \leq N}$ that satisfy*

(i) *compact Fourier support:*

$$\mathrm{supp}(\widehat{f_0^i}), \; \mathrm{supp}(\widehat{f_1^i}) \; \subset \tau_{m_i}^{j_i, k_i},$$

(ii) *boundedness:*

$$2^{\frac{k_i}{2}} |\widehat{f_0^i}|, 2^{\frac{-k_i}{2}} |\widehat{f_1^i}| \leq A |\tau_{m_i}^{j_i, k_i}|^{-\frac{1}{2}},$$

(iii) *closeness:*

$$\|S[u_0, u_1] - \sum_{i=1}^{N} S[f_0^i, f_1^i]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \epsilon,$$

(iv) *orthogonality:*

$$\|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 = \sum_{i=1}^{N} \|(f_0^i, f_1^i)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 + \|(u_0 - \sum_{i=1}^{N} f_0^i, u_1 - \sum_{i=1}^{N} f_1^i)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2.$$

In order to prove Proposition 2.1, the difficulty now is to deal with the upper and lower cones, namely the S_+ and S_- parts. The following lemma helps us to link the Propositions 2.2 and 2.3.

Lemma 2.5. *Let $\{(g^i_{0,n}, g^i_{1,n})_{n\in\mathbb{N}}\}_{1\le i\le N_1}$ be a family of sequences in $\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)$ which satisfies (ii) and (iii) of Proposition 2.3.*

Then there exist $N_2 \le 2N_1$, a family of sequences $\{(r^n_j, \ell^n_j, w^n_j)_{n\in\mathbb{N}}\}_{1\le j\le N_2}$ which satisfies, up to a subsequence,

$$\frac{r^n_j}{r^n_{j'}} + \frac{r^n_{j'}}{r^n_j} + \frac{\ell^n_j}{\ell^n_{j'}} + \frac{\ell^n_{j'}}{\ell^n_j} + \ell^n_j|w^n_j - w^n_{j'}| \underset{n\to\infty}{\to} \infty \qquad \forall\, j \neq j', \tag{2.26}$$

and a family of sequences $\{(P^j_{0,n}, P^j_{1,n})\}_{1\le j\le N_2}$ in $\dot H^{\frac12}\times \dot H^{-\frac12}(\mathbb{R}^d)$, which satisfies, for every j,

$$|\widehat{P^j_{0,n}}|, |\widehat{P^j_{1,n}}| \lesssim \chi_F, \tag{2.27}$$

where $F\subset \mathbb{R}^d\setminus\{0\}$ is a compact set, and such that

$$\sum_{i=1}^{N_1} S[g^i_{0,n}, g^i_{1,n}](x,t) = \sum_{j=1}^{N_2}\left(\frac{r^n_j}{\ell^n_j}\right)^{\frac{d-1}{2}} S[P^j_{0,n}, P^j_{1,n}]\left((T^{\ell^n_j}_{w^n_j})^{-1} r^n_j(x,t)\right). \tag{2.28}$$

Proof. Setting $(r^n_i, \ell^n_i, w^n_i) = (2^{k^n_i}, 2^{j^n_i}, \theta^n_i)$, by (ii) and (iii) of Proposition 2.3, the functions $P^i_{0,n}, P^i_{1,n}$ defined as

$$S[P^i_{0,n}, P^i_{1,n}](x,t) = \left(\frac{\ell^n_i}{r^n_i}\right)^{\frac{d-1}{2}}\Big(S_+[g^i_{0,n}, g^i_{1,n}](T^{\ell^n_i}_{w^n_i}\frac{1}{r^n_i}(x,t)) + S_-[g^i_{0,n}, g^i_{1,n}](T^{\ell^n_i}_{-w^n_i}\frac{1}{r^n_i}(x,t))\Big),$$

satisfy (2.27). We have

$$\begin{aligned}\sum_{i=1}^{N_1} S[g^i_{0,n}, g^i_{1,n}](x,t) = &\sum_{i=1}^{N_1}\left(\frac{r^n_i}{\ell^n_i}\right)^{\frac{d-1}{2}} S_+[P^i_{0,n}, P^i_{1,n}]\left((T^{\ell^n_i}_{w^n_i})^{-1} r^n_i(x,t)\right)\\ &+\left(\frac{r^n_i}{\ell^n_i}\right)^{\frac{d-1}{2}} S_-[P^i_{0,n}, P^i_{1,n}]\left((T^{\ell^n_i}_{-w^n_i})^{-1} r^n_i(x,t)\right),\end{aligned}$$

which is slightly different to (2.28). To overcome this, we redefine the functions $P^i_{0,n}, P^i_{1,n}$. We have that if (r^n_1, ℓ^n_1, w^n_1) is orthogonal in the sense of (2.26), to every $(r^n_i, \ell^n_i, -w^n_i)$, we define

$$\begin{aligned} S[P^1_{0,n}, P^1_{1,n}](x,t) &= \left(\frac{\ell^n_1}{r^n_1}\right)^{\frac{d-1}{2}} S_+[g^1_{0,n}, g^1_{1,n}](T^{\ell^n_1}_{w^n_1}\frac{1}{r^n_1}(x,t)),\\ S[P^2_{0,n}, P^2_{1,n}](x,t) &= \left(\frac{\ell^n_1}{r^n_1}\right)^{\frac{d-1}{2}} S_-[g^1_{0,n}, g^1_{1,n}](T^{\ell^n_1}_{-w^n_1}\frac{1}{r^n_1}(x,t)).\end{aligned}$$

These functions, by (ii) and (iii) of Proposition 2.3, satisfy (2.27).

If instead (r_1^n,ℓ_1^n,w_1^n) is not orthogonal to some $(r_i^n,\ell_i^n,-w_i^n)$ with $1\le i\le N$, then, taking a subsequence we can assume

$$\frac{r_1^n}{r_i^n}+\frac{r_i^n}{r_1^n}+\frac{\ell_1^n}{\ell_i^n}+\frac{\ell_i^n}{\ell_1^n}+\ell_1^n|w_1^n+w_i^n|\le C. \tag{2.29}$$

We define

$$S[P_{0,n}^1,P_{1,n}^1](x,t)=\Big(\frac{\ell_1^n}{r_1^n}\Big)^{\frac{d-1}{2}}\Big(S_+[g_{0,n}^1,g_{1,n}^1](T_{w_1^n}^{\ell_1^n}\frac{1}{r_1^n}(x,t))+S_-[g_{0,n}^i,g_{1,n}^i](T_{w_1^n}^{\ell_1^n}\frac{1}{r_1^n}(x,t))\Big).$$

In this case we have that $P_{0,n}^1,P_{1,n}^1$ is Fourier supported in $K_+\cup K_-$, where

$$K_+:=\big\{\xi\in\mathbb{R}^d:\ \ (\xi,|\xi|)=T_{w_1^n}^{\ell_1^n}\frac{1}{r_1^n}(\rho,|\rho|),\ \ \rho\in\mathcal{T}_1^n\big\},$$
$$K_-:=\big\{\xi\in\mathbb{R}^d:\ \ (\xi,|\xi|)=T_{w_1^n}^{\ell_1^n}\frac{1}{r_1^n}(\rho,-|\rho|),\ \ \rho\in\mathcal{T}_i^n\big\}.$$

By (ii) of Proposition 2.3, K_+ is contained in a compact set that does not contain the origin. Regarding K_-, we can rewrite it as

$$K_-=\big\{\xi\in\mathbb{R}^d:\ \ (\xi,|\xi|)=T_{w_1^n}^{\ell_1^n}(T_{-w_i^n}^{\ell_i^n})^{-1}\frac{r_i^n}{r_1^n}\big(T_{-w_i^n}^{\ell_i^n}\frac{1}{r_i^n}(\rho,-|\rho|)\big),\ \ \rho\in\mathcal{T}_i^n\big\},$$

and by (2.29), we have that for every compact $K\in\mathbb{R}^{d+1}$ which does not contain the origin, the set

$$K'=T_{w_1^n}^{\ell_1^n}(T_{-w_i^n}^{\ell_i^n})^{-1}\frac{r_i^n}{r_1^n}K,$$

is also compact and does not contain the origin.

Again by (2.29), we have

$$|\widehat{P}_{0,n}^1|,|\widehat{P}_{1,n}^1|\le C\left(1+\Big(\frac{\ell_i^n}{\ell_1^n}\Big)^{\frac{d-1}{2}}\Big(\frac{r_1^n}{r_i^n}\Big)^{\frac{d+1}{2}}\right)\lesssim C,$$

where C is the constant of (iii) in Proposition 2.3. Thus the functions $P_{0,n}^1,P_{1,n}^1$ satisfy (2.27).

We observe that there can only exist one index i with $(r_i^n,\ell_i^n,-w_i^n)$ not orthogonal to (r_1^n,ℓ_1^n,w_1^n). Indeed, if there were two indices i, i' with $(r_i^n,\ell_i^n,-w_i^n)$ and $(r_{i'}^n,\ell_{i'}^n,-w_{i'}^n)$ not orthogonal to (r_1^n,ℓ_1^n,w_1^n), then $(r_i^n,\ell_i^n,-w_i^n)$ and $(r_{i'}^n,\ell_{i'}^n,-w_{i'}^n)$ would not be orthogonal aso that (r_i^n,ℓ_i^n,w_i^n) and $(r_{i'}^n,\ell_{i'}^n,w_{i'}^n)$ would not be either, which is a contradiction.

Then it is clear that if we iterate the process for (r_i^n,ℓ_i^n,w_i^n) with $i\le N$, we obtain N_2 functions $\{P_{0,n}^j,P_{1,n}^j\}_{1\le j\le N_2}$ with $N_2\le 2N_1$, which satisfy (2.27), and renaming the w_j^n if necessary, a family of sequences $\{r_j^n,\ell_j^n,w_j^n\}_{1\le j\le N_2}$ which satisfies (2.26). Noting that if (r_1^n, ℓ_1^n, w_1^n) is not orthogonal to $(r_i^n, \ell_i^n, -w_i^n)$, then (r_i^n, ℓ_i^n, w_i^n) is also not orthogonal to $(r_1^n, \ell_1^n, -w_1^n)$, the term

$$\left(\frac{\ell_i^n}{r_i^n}\right)^{\frac{d-1}{2}} \Big(S_+[g_{0,n}^i, g_{1,n}^i](T_{w_i^n}^{\ell_i^n} \frac{1}{r_i^n}(x,t)) + S_-[g_{0,n}^1, g_{1,n}^1](T_{w_i^n}^{\ell_i^n} \frac{1}{r_i^n}(x,t))\Big)$$

will appear in the process as one of the $S[P_{0,n}^j, P_{1,n}^j]$. We therefore obtain (2.28).

□

Proof of Proposition 2.1. By Proposition 2.3, for every $\epsilon > 0$ and for every n, there exists a family of functions $\big\{(g_{0,n}^i, g_{1,n}^i)\big\}_{1\le i\le N_1}$ and $(\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1})$ such that

$$S[u_{0,n}, u_{1,n}](x,t) = \sum_{i=1}^{N_1} S[g_{0,n}^i, g_{1,n}^i](x,t) + S[\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1}](x,t), \tag{2.30}$$

with

$$\limsup_{n\to\infty} \left\| S[\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1}] \right\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \frac{\epsilon}{2}, \tag{2.31}$$

satisfying (ii) and (iii) of Proposition 2.3 and

$$\|(u_{0,n}, u_{1,n})\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} = \sum_{i=1}^{N_1} \|(g_{0,n}^i, g_{1,n}^i)\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} + \|(\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1})\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}}. \tag{2.32}$$

Now, by Lemma 2.5, we can write

$$S[u_{0,n}, u_{1,n}](x,t) = \sum_{j=1}^{N_2} \left(\frac{r_j^n}{\ell_j^n}\right)^{\frac{d-1}{2}} S[P_{0,n}^j, P_{1,n}^j] \left((T_{w_j^n}^{\ell_j^n})^{-1} r_j^n(x,t)\right) + S[\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1}](x,t),$$

where $N_2 \le 2N_1$, $\{(r_j^n, \ell_j^n, w_j^n)\}_{1\le j\le N_2}$ is a family of sequences which obeys (2.26), and the family of sequences $\{\big(P_{0,n}^j, P_{1,n}^j\big)_{n\in\mathbb{N}}\}_{1\le j\le N_2}$ satisfies (2.27).

By Proposition 2.2 applied to $\big(P_{0,n}^j, P_{1,n}^j\big)_{n\in\mathbb{N}}$ for each j, we have the decomposition

$$\begin{aligned} S[u_{0,n}, u_{1,n}](x,t) &= \sum_{j=1}^{N_2}\sum_{\alpha=1}^{A} \left(\frac{r_j^n}{\ell_j^n}\right)^{\frac{d-1}{2}} S[\phi_0^{j,\alpha}, \phi_1^{j,\alpha}] \left((T_{w_j^n}^{\ell_j^n})^{-1} r_j^n(x - x_{j,\alpha}^n, t - t_{j,\alpha}^n)\right) \\ &\quad + \sum_{j=1}^{N_2} \left(\frac{r_j^n}{\ell_j^n}\right)^{\frac{d-1}{2}} S[\mathtt{P}_{0,j,n}^A, \mathtt{P}_{1,j,n}^A] \left((T_{w_j^n}^{\ell_j^n})^{-1} r_j^n(x,t)\right) + S[\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1}](x,t) \\ &:= \sum_{j=1}^{N_2}\sum_{\alpha=1}^{A} \Gamma_{(j,\alpha)}^n S[\phi_0^{j,\alpha}, \phi_1^{j,\alpha}](x,t) \\ &\quad + \sum_{j=1}^{N_2} \left(\frac{r_j^n}{\ell_j^n}\right)^{\frac{d-1}{2}} S[\mathtt{P}_{0,j,n}^A, \mathtt{P}_{1,j,n}^A] \left((T_{w_j^n}^{\ell_j^n})^{-1} r_j^n(x,t)\right) + S[\mathtt{Q}_{0,n}^{N_1}, \mathtt{Q}_{1,n}^{N_1}](x,t), \end{aligned}$$

where $(x^n_{j,\alpha}, t^n_{j,\alpha}) = T^{\ell^n_j}_{w^n_j} \frac{(y^n_{j,\alpha}, s^n_{j,\alpha})}{r^n_i}$, and with the sequences associated to $\Gamma^n_{(j,\alpha)}$ being $(r^n_j, \ell^n_j, w^n_j, x^n_{j,\alpha}, t^n_{j,\alpha})$. Moreover

$$|y^n_{j,\alpha} - y^n_{j,\alpha'}| + |s^n_{j,\alpha} - s^n_{j,\alpha'}| \underset{n\to\infty}{\longrightarrow} +\infty, \quad \text{for every } (j,\alpha) \neq (j,\alpha'), \tag{2.33}$$

and for each j,

$$\lim_{A\to\infty} \lim_{n\to\infty} \|S[\mathtt{P}^A_{0,j,n}, \mathtt{P}^A_{1,j,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = 0.$$

We choose A so that

$$\lim_{n\to\infty} \|S[\mathtt{P}^A_{0,j,n}, \mathtt{P}^A_{1,j,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \frac{\epsilon}{2N_2}, \tag{2.34}$$

for every $1 \leq j \leq N_2$. Therefore if we denote

$$S[R_{0,n}, R_{1,n}](x,t) = \sum_{j=1}^{N_2} \left(\frac{r^n_j}{\ell^n_j}\right)^{\frac{d-1}{2}} S[\mathtt{P}^A_{0,j,n}, \mathtt{P}^A_{1,j,n}]\left((T^{\ell^n_j}_{w^n_j})^{-1} r^n_j(x,t)\right) + S[\mathtt{Q}^{N_1}_{0,n}, \mathtt{Q}^{N_1}_{1,n}](x,t),$$

we have, relabeling the pairs (j,α) and taking $N = A \cdot N_2$,

$$S[u_{0,n}, u_{1,n}](x,t) = \sum_{j=1}^{N} \Gamma^n_j S[\phi^j_0, \phi^j_1](x,t) + S[R_{0,n}, R_{1,n}](x,t), \tag{2.35}$$

such that from (2.31) and (2.34)

$$\limsup_{n\to\infty} \|S[R_{0,n}, R_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \epsilon, \tag{2.36}$$

from (2.15) and (2.32)

$$\begin{aligned} &\|(u_{0,n}, u_{1,n})\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} \\ &= \sum_{j=1}^{N} \|(\phi^j_0, \phi^j_1)\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} + \|(R_{0,n}, R_{1,n})\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} + o(1),\ n \to \infty, \end{aligned} \tag{2.37}$$

and by (2.33) and (2.26), we can take a subsequence which is orthogonal. Now, by Lemma 2.1, taking a subsequence, we have

$$\limsup_{n\to\infty} \Big\|\sum_{j=1}^{N} \Gamma^n_j S[\phi^j_0, \phi^j_1]\Big\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = \sum_{j=1}^{N} \|S[\phi^j_0, \phi^j_1]\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})},$$

so that taking $\epsilon \leq \frac{K}{2}$, by (2.35) and (2.36),

$$2\sum_{j=1}^{N} \|S[\phi^j_0, \phi^j_1]\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq K^{2\frac{d+1}{d-1}}.$$

By Hölder, the Strichartz inequality (2.2), (2.37) and the hypothesis,

$$\begin{aligned}\sum_{j=1}^{N}\|S[\phi_0^j,\,\phi_1^j]\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} &\lesssim \sup_j \|S[\phi_0^j,\phi_1^j]\|^{\frac{4}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\sum_{j=1}^{N}\|(\phi_0^j,\phi_1^j)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}\\ &\lesssim \sup_j \|S[\phi_0^j,\phi_1^j]\|^{\frac{4}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\limsup_{n\to\infty}\|(u_{0,n},u_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}\\ &\lesssim \sup_j \|S[\phi_0^j,\phi_1^j]\|^{\frac{4}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}M^2,\end{aligned}$$

so that, there exists j_0 such that

$$\|S[\phi_0^{j_0},\phi_1^{j_0}]\|^{\frac{4}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\gtrsim \frac{K^{2\frac{d+1}{d-1}}}{M^2}.$$

Taking the inverse transformation $(\Gamma^n_{j_0})^{-1}$, we get from (2.35),

$$\begin{aligned}&(\Gamma^n_{j_0})^{-1}S[u_{0,n},u_{1,n}]\\ &=S[\phi_0^{j_0},\phi_1^{j_0}]+\sum_{j=1,\,j\neq j_0}^{N}(\Gamma^n_{j_0})^{-1}\Gamma^n_j S[\phi_0^j,\phi_1^j]+(\Gamma^n_{j_0})^{-1}S[R_{0,n},R_{1,n}].\end{aligned}$$

By Lemma 2.2, we have for every $j\neq j_0$,

$$(\Gamma^n_{j_0})^{-1}\Gamma^n_j S[\phi_0^j,\phi_1^j]\underset{n\to\infty}{\rightharpoonup}0$$

and therefore

$$(\Gamma^n_{j_0})^{-1}S[u_{0,n},u_{1,n}]\underset{n\to\infty}{\rightharpoonup}\mathtt{U}=S[\phi_0^{j_0},\phi_1^{j_0}]+\mathtt{W},$$

where $\mathtt{W}$ is the weak limit of $(\Gamma^n_{j_0})^{-1}S[R_{0,n},R_{1,n}]$. Now, as

$$\|\mathtt{W}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\le\limsup_{n\to\infty}\|(\Gamma^n_{j_0})^{-1}S[R_{0,n},R_{1,n}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}<\epsilon,$$

we conclude that

$$\|\mathtt{U}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\gtrsim\frac{K^{2\frac{d+1}{d-1}}}{M^2}$$

again taking ϵ sufficiently small, and the proof is complete.

$\square$

2.3. Proof of Theorem 2.1

We require the following lemma, which is a simplification of Proposition 2.2, with a weaker hypothesis, but with a weaker smallness of the remainder property and considered together with the space-time translations, the Lorentz symmetries and rescalings.

Letting $(\mathbf{u_0},\mathbf{u_1})=(u_{0,n},u_{1,n})_n$ be a bounded sequence in $\dot H^{\frac12}\times\dot H^{-\frac12}(\mathbb{R}^d)$, we define the set $\mathcal{W}(\mathbf{u_0},\mathbf{u_1})$ by

$$\mathcal{W}(\mathbf{u_0},\mathbf{u_1})=\left\{\begin{array}{c|c} \begin{array}{l}(\phi_0,\phi_1)\\ \in \dot H^{\frac12}\times\dot H^{-\frac12}\end{array} & \begin{array}{c}\text{there exist transformations } \Gamma^n\\ \text{such that, up to a subsequence :}\\ (\Gamma^n)^{-1}S[u_{0,n},u_{1,n}](x,0)\underset{n\to\infty}{\rightharpoonup}\phi_0 \text{ weakly in } \dot H^{\frac12}\\ \partial_t(\Gamma^n)^{-1}S[u_{0,n},u_{1,n}](x,0)\underset{n\to\infty}{\rightharpoonup}\phi_1 \text{ weakly in } \dot H^{-\frac12}.\end{array}\end{array}\right\},$$

and write

$$\mu(\mathbf{u_0},\mathbf{u_1})=\sup\left\{\|(\phi_0,\phi_1)\|_{\dot H^{\frac12}\times\dot H^{-\frac12}}\quad;\quad(\phi_1,\phi_2)\in\mathcal{W}(\mathbf{u_0},\mathbf{u_1})\right\}.$$

Lemma 2.6. *Let $d\geq 2$ and $(u_{0,n},u_{1,n})_n$ be a bounded sequence in $\dot H^{\frac12}\times\dot H^{-\frac12}(\mathbb{R}^d)$ Then, there exist a subequence (still denoted $(u_{0,n},u_{1,n})_n$), a sequence $(\phi_0^\alpha,\phi_1^\alpha)_\alpha$, and a family of orthogonal sequences $\{(r_\alpha^n,\ell_\alpha^n,w_\alpha^n,x_\alpha^n,t_\alpha^n)_{n\in\mathbb{N}}\}_\alpha$ in $\mathbb{R}^+\times[1,\infty)\times\mathbb{S}^{d-1}\times\mathbb{R}^d\times\mathbb{R}$, $\alpha\in\mathbb{N}$ such that*

$$S[u_{0,n},u_{1,n}](x,t)=\sum_{\alpha=1}^N\Gamma_\alpha^nS[\phi_0^\alpha,\phi_1^\alpha](x,t)+S[R_{0,n}^N,R_{1,n}^N](x,t),\tag{2.38}$$

with

$$\mu(\mathbf{R_0^N},\mathbf{R_1^N})\underset{N\to+\infty}{\longrightarrow}0\quad\textit{where}\quad(\mathbf{R_0^N},\mathbf{R_1^N})=(R_{0,n}^N,R_{1,n}^N)_n,\tag{2.39}$$

and the orthogonality property
(2.40)

$$\|(u_{0,n},u_{1,n})\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}=\sum_{\alpha=1}^N\|(\phi_0^\alpha,\phi_1^\alpha)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}+\|(R_{0,n}^N,R_{1,n}^N)\|^2_{\dot H^{\frac12}\times\dot H^{-\frac12}}+o(1)\ \textit{as}\ n\to\infty\,.$$

Proof. The proof is analogous to that of Proposition 2.2, where we just have to ensure that the sequences are orthogonal. We include the argument for completeness.

We extract the functions $\phi_0^\alpha,\phi_1^\alpha$ recursively. If $\mu(\mathbf{u_0},\mathbf{u_1})=0$, then we can take $\phi_0^\alpha\equiv 0$, $\phi_1^\alpha\equiv 0$ for all α and we are done. Otherwise, there exists $(\phi_0^1,\phi_1^1)\in\mathcal{W}(\mathbf{u_0},\mathbf{u_1})$ such that

$$\|(\phi_0^1,\phi_1^1)\|_{\dot H^{\frac12}\times\dot H^{-\frac12}}\geq\frac12\mu(\mathbf{u_0},\mathbf{u_1})>0.$$

By the definition, we can choose a sequence $(r_1^n, \ell_1^n, w_1^n, x_1^n, t_1^n)$ in $\mathbb{R}^+ \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$ such that, up to extracting a subsequence, we have:

$$(\Gamma_1^n)^{-1} S[u_{0,n}, u_{1,n}](x,0) \underset{n\to\infty}{\rightharpoonup} \phi_0^1 \quad \text{weakly in } \dot{H}^{\frac{1}{2}},$$
$$\partial_t(\Gamma_1^n)^{-1} S[u_{0,n}, u_{1,n}](x,0) \underset{n\to\infty}{\rightharpoonup} \phi_1^1 \quad \text{weakly in } \dot{H}^{-\frac{1}{2}}.$$

We set

$$R_{0,n}^1(x) := u_{0,n}(x) - \Gamma_1^n S[\phi_0^1, \phi_1^1](x,0),$$
$$R_{1,n}^1(x) := u_{1,n}(x) - \partial_t \Gamma_1^n S[\phi_0^1, \phi_1^1](x,0),$$

so that

$$(\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](x,0) \underset{n\to\infty}{\rightharpoonup} 0 \quad \text{and} \quad \partial_t(\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](x,0) \underset{n\to\infty}{\rightharpoonup} 0. \tag{2.41}$$

Now,

$$\begin{aligned}
&\|u_{0,n}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|u_{1,n}\|_{\dot{H}^{-\frac{1}{2}}}^2 = \left\|\Gamma_1^n S[\phi_0^1, \phi_1^1](\cdot,0)\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\partial_t \Gamma_1^n S[\phi_0^1, \phi_1^1](\cdot,0)\right\|_{\dot{H}^{-\frac{1}{2}}}^2 \\
&+ \left\|R_{0,n}^1\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|R_{1,n}^1\right\|_{\dot{H}^{-\frac{1}{2}}}^2 + 2\left\langle R_{0,n}^1, \Gamma_1^n S[\phi_0^1, \phi_1^1](\cdot,0)\right\rangle_{\dot{H}^{\frac{1}{2}}} + 2\left\langle R_{1,n}^1, \partial_t \Gamma_1^n S[\phi_0^1, \phi_1^1](\cdot,0)\right\rangle_{\dot{H}^{-\frac{1}{2}}} \\
&= \left\|\Gamma_1^n S[\phi_0^1, \phi_1^1](\cdot,0)\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\partial_t \Gamma_1^n S[\phi_0^1, \phi_1^1](\cdot,0)\right\|_{\dot{H}^{-\frac{1}{2}}}^2 + \left\|R_{0,n}^1\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|R_{1,n}^1\right\|_{\dot{H}^{-\frac{1}{2}}}^2 \\
&+ 2\left\langle \phi_0^1, (\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0)\right\rangle_{\dot{H}^{\frac{1}{2}}} + 2\left\langle \phi_1^1, \partial_t(\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0)\right\rangle_{\dot{H}^{-\frac{1}{2}}} \\
&= \|\phi_0^1\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\phi_1^1\|_{\dot{H}^{-\frac{1}{2}}}^2 + \left\|R_{0,n}^1\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|R_{1,n}^1\right\|_{\dot{H}^{-\frac{1}{2}}}^2 \\
&+ 2\left\langle \phi_0^1, (\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0)\right\rangle_{\dot{H}^{\frac{1}{2}}} + 2\left\langle \phi_1^1, \partial_t(\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0)\right\rangle_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}$$

Therefore, by (2.41), we have

$$\left\|(u_{0,n}, u_{1,n})\right\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 = \left\|(\phi_0^1, \phi_1^1)\right\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 + \left\|(R_{0,n}^1, R_{1,n}^1)\right\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^2 + o(1).$$

We repeat the above process replacing $(u_{0,n}, u_{1,n})_n$ with $(R_{0,n}^1, R_{1,n}^1)_n$. If $\mu(\mathbf{R_0^1}, \mathbf{R_1^1}) > 0$, we obtain ϕ_0^2, ϕ_1^2, $(r_2^n, \ell_2^n, w_2^n, x_2^n, t_2^n)$ and $(R_{0,n}^2, R_{1,n}^2)_n$.

To prove that the orthogonality between Γ_1^n and Γ_2^n we suppose otherwise. For every pair $(h_1, h_2) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$,

$$\begin{aligned}
&\langle (\Gamma_2^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0), h_1\rangle_{\dot{H}^{\frac{1}{2}}} + \langle \partial_t(\Gamma_2^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0), h_2\rangle_{\dot{H}^{-\frac{1}{2}}} \\
&\quad = \langle (\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0), (\Gamma_1^n)^{-1}\Gamma_2^n S[h_1, h_2](\cdot,0)\rangle_{\dot{H}^{\frac{1}{2}}} \\
&\qquad + \langle \partial_t(\Gamma_1^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0), \partial_t(\Gamma_1^n)^{-1}\Gamma_2^n S[h_1, h_2](\cdot,0)\rangle_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}$$

Thus, by (2.41) and the strong convergence of $(\Gamma_1^n)^{-1}\Gamma_2^n S[h_1, h_2](\cdot,0) \to \Gamma S[h_1, h_2](\cdot,0)$ and $\partial_t(\Gamma_1^n)^{-1}\Gamma_2^n S[h_1, h_2](\cdot,0) \to \partial_t \Gamma S[h_1, h_2](\cdot,0)$, where Γ is isometric in $\dot{H}^{\frac{1}{2}}$ (see the proof of Lemma 2.1 for more details) we obtain

$$\langle (\Gamma_2^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0), h_1 \rangle_{\dot{H}^{\frac{1}{2}}} + \langle \partial_t (\Gamma_2^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0), h_2 \rangle_{\dot{H}^{-\frac{1}{2}}} \to 0.$$

Recalling that $(\Gamma_2^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0) \rightharpoonup \phi_0^2$, $\partial_t (\Gamma_2^n)^{-1} S[R_{0,n}^1, R_{1,n}^1](\cdot,0) \rightharpoonup \phi_1^2$, the uniqueness of weak limits would imply that $\phi_0^2 = 0$ and $\phi_1^2 = 0$, and therefore $\mu(\mathbf{R_0^1}, \mathbf{R_1^1}) = 0$, which gives a contradiction. Iterating the process we get $(\phi_0^\alpha, \phi_1^\alpha)_\alpha$ and $(r_\alpha^n, \ell_\alpha^n, w_\alpha^n, x_\alpha^n, t_\alpha^n)_\alpha$ satisfying (2.38) and (2.40). It remains to prove (2.39) but this is done exactly as in Proposition 2.2.

□

Proof of Theorem 2.1. We apply Lemma 2.6, so that it remains to prove

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S[R_{0,n}^N, R_{1,n}^N]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = 0.$$

We suppose for a contradiction that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S[R_{0,n}^N, R_{1,n}^N]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \neq 0.$$

Then we could find a subsequence $N_k \to \infty$, and $K > 0$, such that for every $k \in \mathbb{N}$,

$$\limsup_{n\to\infty} \|S[R_{0,n}^{N_k}, R_{1,n}^{N_k}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq K.$$

On the other hand we have by (2.40),

$$\limsup_{n\to\infty} \|(R_{0,n}^{N_k}, R_{1,n}^{N_k})\|^2_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \leq \limsup_{n\to\infty} \|(u_{0,n}, u_{1,n})\|^2_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} =: M.$$

We will use these to violate Lemma 2.6.

By Proposition 2.1, for every $k \in \mathbb{N}$, there exists a transformation that we denote Γ_k^n, such that

$$(\Gamma_k^n)^{-1} S[R_{0,n}^{N_k}, R_{1,n}^{N_k}] \underset{n\to\infty}{\rightharpoonup} S[\mathtt{R}_0^{N_k}, \mathtt{R}_1^{N_k}] \quad \text{weakly in } L^{2\frac{d+1}{d-1}}, \tag{2.42}$$

with

$$\|S[\mathtt{R}_0^{N_k}, \mathtt{R}_1^{N_k}]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq C(K, M) > 0.$$

By the Strichartz inequality (2.2), we get

$$\|(\mathtt{R}_0^{N_k}, \mathtt{R}_1^{N_k})\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}} \gtrsim C(K, M) > 0. \tag{2.43}$$

Now, Lemma 2.3 says that (2.42) is equivalent to

$$\big((\Gamma_k^n)^{-1} S[R_{0,n}^{N_k}, R_{1,n}^{N_k}](\cdot,0), (\Gamma_k^n)^{-1} \partial_t S[R_{0,n}^{N_k}, R_{1,n}^{N_k}](\cdot,0)\big) = (\mathtt{R}_{0,n}^{N_k}, \mathtt{R}_{1,n}^{N_k}) \underset{n\to\infty}{\rightharpoonup} (\mathtt{R}_0^{N_k}, \mathtt{R}_1^{N_k}),$$

and we deduce that $(\mathtt{R}_0^{\mathtt{N}_k}, \mathtt{R}_1^{\mathtt{N}_k}) \in \mathcal{W}(\mathbf{R_0^{N_k}}, \mathbf{R_1^{N_k}})$ for every $k \in \mathbb{N}$. Then by (2.43),

$$\mu(\mathbf{R_0^{N_k}}, \mathbf{R_1^{N_k}}) \gtrsim C(K, M),$$

which contradicts (2.39), and we are done.

$\square$

2.4. Orthogonality

It remains to prove Lemmas 2.1 and 2.2. We will require the following lemma due to Bahouri and Gérard [1].

Lemma 2.7. [1] *For all $p \in [2, \infty)$,*

$$\Big| \big| \sum_{j=1}^{N} a_j \big|^p - \sum_{j=1}^{N} |a_j|^p \Big| \leq C_N \sum_{j \neq k} |a_j| |a_k|^{p-1}.$$

We introduce also the following definition

Definition 2.1. *Two sequences $(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{n \in \mathbb{N}}$, $(r_k^n, \ell_k^n, w_k^n, x_k^n, t_k^n)_{n \in \mathbb{N}}$ are in balance if*

$$\frac{r_j^n \ell_k^n}{r_k^n \ell_j^n} + \frac{r_k^n \ell_j^n}{r_j^n \ell_k^n} \underset{n \to \infty}{\not\to} \infty. \tag{2.44}$$

Proof of Lemma 2.1. We can assume that $\|S[\phi_0^j, \phi_1^j]\|_{L_{t,x}^{2\frac{d+1}{d-1}}} \leq 1$ for $1 \leq j \leq N$. Using Lemma 2.7 and that the transformations Γ_j^n conserve the $L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})$,

$$\begin{aligned}
&\Big| \| \sum_{j=1}^{N} \Gamma_j^n S[\phi_0^j, \phi_1^j] \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} - \sum_{j=1}^{N} \| S[\phi_0^j, \phi_1^j] \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} \Big| \\
&\qquad = \Big| \| \sum_{j=1}^{N} \Gamma_j^n S[\phi_0^j, \phi_1^j] \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} - \sum_{j=1}^{N} \| \Gamma_j^n S[\phi_0^j, \phi_1^j] \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} \Big| \\
&\qquad \leq \iint \Big| \big| \sum_{j=1}^{N} \Gamma_j^n S[\phi_0^j, \phi_1^j](x,t) \big|^{2\frac{d+1}{d-1}} - \sum_{j=1}^{N} \big| \Gamma_j^n S[\phi_0^j, \phi_1^j](x,t) \big|^{2\frac{d+1}{d-1}} \Big| dx dt \\
&\qquad \leq C_N \iint \sum_{j \neq k} \big| \Gamma_j^n S[\phi_0^j, \phi_1^j](x,t) \big| \big| \Gamma_k^n S[\phi_0^k, \phi_1^k](x,t) \big|^{\frac{d+3}{d-1}} dx dt.
\end{aligned}$$

For fixed $j \neq k$, we will prove that

$$\iint \left|\Gamma_j^n S[\phi_0^j, \phi_1^j](x,t)\right|\left|\Gamma_k^n S[\phi_0^k, \phi_1^k](x,t)\right|^{\frac{d+3}{d-1}} dxdt \underset{n\to\infty}{\longrightarrow} 0.$$

For $R > 0$, we define the sets

$$\Lambda_R^{j,n} := \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} \ : \quad \left|(T_{w_j^n}^{\ell_j^n})^{-1} r_j^n (x - x_j^n, t - t_j^n)\right| < R\right\}$$

and $\Lambda_R^{k,n}$ in the same way. We split

$$\mathbb{R}^d \times \mathbb{R} = \left((\mathbb{R}^d \times \mathbb{R}) \setminus \Lambda_R^{j,n}\right) \bigcup \left((\mathbb{R}^d \times \mathbb{R}) \setminus \Lambda_R^{k,n}\right) \bigcup (\Lambda_R^{j,n} \cap \Lambda_R^{k,n}),$$

and estimate the integral in these regions.

For every $\epsilon > 0$, there exists an R_0 sufficiently big for which

$$\int_{\left((\mathbb{R}^d\times\mathbb{R})\setminus B_{R_0}\right)} \left|S[\phi_0^j, \phi_1^j]\right|^{2\frac{d+1}{d-1}} dxdt < \epsilon, \quad \int_{\left((\mathbb{R}^d\times\mathbb{R})\setminus B_{R_0}\right)} \left|S[\phi_0^k, \phi_1^k]\right|^{2\frac{d+1}{d-1}} dxdt < \epsilon,$$

so that by Hölder's inequality and a change of variables,

$$\begin{aligned}
&\int_{\left((\mathbb{R}^d\times\mathbb{R})\setminus \Lambda_{R_0}^{j,n}\right)} \left|\Gamma_j^n S[\phi_0^j, \phi_1^j](x,t)\right|\left|\Gamma_k^n S[\phi_0^k, \phi_1^k](x,t)\right|^{\frac{d+3}{d-1}} dxdt \\
&\leq \Big(\int_{\left((\mathbb{R}^d\times\mathbb{R})\setminus B_{R_0}\right)} \left|S[\phi_0^j, \phi_1^j]\right|^{2\frac{d+1}{d-1}} dxdt\Big)^{\frac{d-1}{2(d+1)}} \Big(\iint \left|S[\phi_0^k, \phi_1^k]\right|^{2\frac{d+1}{d-1}} dxdt\Big)^{\frac{d+3}{2(d+1)}} < \epsilon,
\end{aligned}$$

and in the same way

$$\int_{\left((\mathbb{R}^d\times\mathbb{R})\setminus \Lambda_{R_0}^{k,n}\right)} \left|\Gamma_j^n S[\phi_0^j, \phi_1^j](x,t)\right|\left|\Gamma_k^n S[\phi_0^k, \phi_1^k](x,t)\right|^{\frac{d+3}{d-1}} dxdt < \epsilon,$$

so we have reduced the problem to show that there exists n_0, such that if $n > n_0$,

$$\int_{(\Lambda_{R_0}^{j,n} \cap \Lambda_{R_0}^{k,n})} \left|\Gamma_j^n S[\phi_0^j, \phi_1^j](x,t)\right|\left|\Gamma_k^n S[\phi_0^k, \phi_1^k](x,t)\right|^{\frac{d+3}{d-1}} dxdt < \epsilon.$$

We define $0 < M < \infty$ by

$$\|S[\phi_0^j, \phi_1^j]\chi_{|S[\phi_0^j,\phi_1^j]|>M}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}, \|S[\phi_0^k, \phi_1^k]\chi_{|S[\phi_0^k,\phi_1^k]|>M}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \frac{\epsilon}{2}.$$

Splitting the integral and by Hölder's inequality

$$\begin{aligned}
&\int_{(\Lambda^{j,n}_{R_0}\cap\Lambda^{k,n}_{R_0})} \big|\Gamma^n_j S[\phi^j_0,\phi^j_1](x,t)\big|\big|\Gamma^n_k S[\phi^k_0,\phi^k_1](x,t)\big|^{\frac{d+3}{d-1}}dxdt\\
\le &\int_{(\Lambda^{j,n}_{R_0}\cap\Lambda^{k,n}_{R_0})} \big|\Gamma^n_j S[\phi^j_0,\phi^j_1](x,t)\big|\big|\Gamma^n_k S[\phi^k_0,\phi^k_1](x,t)\big|^{\frac{d+3}{d-1}}\\
&\qquad\qquad \chi_{\left\{(\frac{\ell^n_k}{r^n_k})^{\frac{d-1}{2}}|\Gamma^n_k S[\phi^k_0,\phi^k_1]|>M\right\}\bigcup\left\{(\frac{\ell^n_j}{r^n_j})^{\frac{d-1}{2}}|\Gamma^n_j S[\phi^j_0,\phi^j_1]|>M\right\}}(x,t)dxdt\\
&+M^{2\frac{d+1}{d-1}}(\frac{r^n_j}{\ell^n_j})^{\frac{d-1}{2}}(\frac{r^n_k}{\ell^n_k})^{\frac{d+3}{2}}\iint\chi_{\Lambda^{j,n}_{R_0}}(x,t)\chi_{\Lambda^{k,n}_{R_0}}(x,t)dxdt\\
\le &\|\Gamma^n_j S[\phi^j_0,\phi^j_1]\chi_{(\frac{\ell^n_j}{r^n_j})^{\frac{d-1}{2}}|\Gamma^n_j S[\phi^j_0,\phi^j_1]|>M}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\|\Gamma^n_k S[\phi^k_0,\phi^k_1]\|^{\frac{d+3}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\
&+\ \|\Gamma^n_j S[\phi^j_0,\phi^j_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\|\Gamma^n_k S[\phi^k_0,\phi^k_1]\chi_{(\frac{\ell^n_k}{r^n_k})^{\frac{d-1}{2}}|\Gamma^n_k S[\phi^k_0,\phi^k_1]|>M}\|^{\frac{d+3}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}\\
&+M^{2\frac{d+1}{d-1}}(\frac{r^n_j}{\ell^n_j})^{\frac{d-1}{2}}(\frac{r^n_k}{\ell^n_k})^{\frac{d+3}{2}}\iint\chi_{\Lambda^{j,n}_{R_0}}(x,t)\chi_{\Lambda^{k,n}_{R_0}}(x,t)dxdt\\
&<\frac{\epsilon}{2}+M^{2\frac{d+1}{d-1}}(\frac{r^n_j}{\ell^n_j})^{\frac{d-1}{2}}(\frac{r^n_k}{\ell^n_k})^{\frac{d+3}{2}}\iint\chi_{\Lambda^{j,n}_{R_0}}(x,t)\chi_{\Lambda^{k,n}_{R_0}}(x,t)dxdt.
\end{aligned}$$

By a change of variables, it will suffice to prove that

$$\begin{aligned}
I_n := (\frac{r^n_j}{\ell^n_j})^{\frac{d-1}{2}}(\frac{r^n_k}{\ell^n_k})^{\frac{d+3}{2}}\iint &\chi_{B_1}((T^{\ell^n_j}_{w^n_j})^{-1}r^n_j(x-x^n_j,t-t^n_j))\\
&\chi_{B_1}((T^{\ell^n_k}_{w^n_k})^{-1}r^n_k(x-x^n_k,t-t^n_k))\ dxdt
\end{aligned} \tag{2.45}$$

is dominated by $\lesssim \epsilon$, where B_1 is the unit ball in $\mathbb{R}^{d+1}$. We now, separate cases according to the nature of the orthogonality on the sequences.

Case 1: The sequences satisfy the Rescaling or Lorentz property, and are not in balance.

Using the change of variables $(x,t)\mapsto T^{\ell^n_j}_{w^n_j}\dfrac{(x,t)}{r^n_j}+(x^n_j,t^n_j)$, we get

$$I_n=\Big(\frac{r^n_k}{\ell^n_k}\frac{\ell^n_j}{r^n_j}\Big)^{\frac{d+3}{2}}\iint\chi_{B_1}(x,t)\chi_{B_1}\Big((T^{\ell^n_k}_{w^n_k})^{-1}T^{\ell^n_j}_{w^n_j}\frac{r^n_k}{r^n_j}(x,t)+(T^{\ell^n_k}_{w^n_k})^{-1}r^n_k(x^n_j-x^n_k,t^n_j-t^n_k)\Big)dxdt,$$

which can be written as

$$I_n=\Big(\frac{r^n_k}{\ell^n_k}\frac{\ell^n_j}{r^n_j}\Big)^{\frac{d+3}{2}}\Big|B_1\bigcap\Big((T^{\ell^n_j}_{w^n_j})^{-1}T^{\ell^n_k}_{w^n_k}\frac{r^n_j}{r^n_k}\big(B_1-(T^{\ell^n_k}_{w^n_k})^{-1}r^n_k(x^n_j-x^n_k,t^n_j-t^n_k)\big)\Big)\Big|, \tag{2.46}$$

which is bounded by

$$\Big(\frac{r_k^n}{\ell_k^n}\frac{\ell_j^n}{r_j^n}\Big)^{\frac{d+3}{2}}|B_1|.$$

If instead we use the change of variables $(x,t)\mapsto T_{w_k^n}^{\ell_k^n}\dfrac{(x,t)}{r_k^n}+(x_k^n,t_k^n)$, we get

$$I_n=\Big(\frac{r_j^n}{\ell_j^n}\frac{\ell_k^n}{r_k^n}\Big)^{\frac{d-1}{2}}\Big|B_1\bigcap\Big((T_{w_k^n}^{\ell_k^n})^{-1}T_{w_j^n}^{\ell_j^n}\frac{r_k^n}{r_j^n}\big(B_1-(T_{w_j^n}^{\ell_j^n})^{-1}r_j^n(x_k^n-x_j^n,t_k^n-t_j^n)\big)\Big)\Big|, \tag{2.47}$$

which in this case is bounded by

$$\Big(\frac{r_j^n}{\ell_j^n}\frac{\ell_k^n}{r_k^n}\Big)^{\frac{d-1}{2}}|B_1|.$$

Putting it together, we have

$$I_n\le\min\Big((\frac{r_k^n}{r_j^n}\frac{\ell_j^n}{\ell_k^n})^{\frac{d+3}{2}},(\frac{r_j^n}{r_k^n}\frac{\ell_k^n}{\ell_j^n})^{\frac{d-1}{2}}\Big).$$

As (2.44) does not hold, we conclude the result.

Case 2: The sequences satisfy the Rescaling or Lorentz property, and are in balance.

From (2.46), we can bound I_n by

$$I_n\le\sup_{(y_0,s_0),(y_1,s_1)\in\mathbb{R}^d\times\mathbb{R}}\Big(\frac{r_k^n}{\ell_k^n}\frac{\ell_j^n}{r_j^n}\Big)^{\frac{d+3}{2}}\Big|\Big(B_1+(y_0,s_0)\Big)\bigcap\Big((T_{w_j^n}^{\ell_j^n})^{-1}T_{w_k^n}^{\ell_k^n}\frac{r_j^n}{r_k^n}B_1+(y_1,s_1)\Big)\Big|.$$

It is easy to see that we have the maximal intersection when $w_k^n=w_j^n$,

$$I_n\le\Big(\frac{r_k^n}{\ell_k^n}\frac{\ell_j^n}{r_j^n}\Big)^{\frac{d+3}{2}}\Big|\Big(B_1+(y_0^n,s_0^n)\Big)\bigcap\Big(T_{w_k^n}^{\frac{\ell_k^n}{\ell_j^n}}\frac{r_j^n}{r_k^n}B_1+(y_1^n,s_1^n)\Big)\Big|.$$

We have that $T_{w_k^n}^{\frac{\ell_k^n}{\ell_j^n}}\dfrac{r_j^n}{r_k^n}B_1$ is contained in a parallelepiped $P_{i,k}^n$ of dimensions $\dfrac{r_j^n(\ell_k^n)^2}{r_k^n(\ell_j^n)^2}\times\dfrac{r_j^n}{r_k^n}\times\dfrac{\ell_k^n r_j^n}{r_k^n\ell_j^n}\times\cdots\times\dfrac{\ell_k^n r_j^n}{r_k^n\ell_j^n}$, and we can conclude that

$$I_n\le\Big|\Big(B_1+(y_0^n,s_0^n)\Big)\bigcap\Big(P_{i,k}^n+(y_1^n,s_1^n)\Big)\Big|\underset{n\to\infty}{\longrightarrow}0,$$

because if the sequence is in balance, then (2.4) implies (2.3) and viceversa.

Case 3: The sequences satisfy the Angular property.

We have that $\chi_{B_1}((T^{\ell_j^n}_{w_j^n})^{-1}r_j^n(x-x_j^n,t-t_j^n))$ is supported in a parallelepiped $P_1 := \frac{(\ell_j^n)^2}{r_j^n} \times \frac{1}{r_j^n} \times \frac{\ell_j^n}{r_j^n} \times \cdots \times \frac{\ell_j^n}{r_j^n}$ with the smallest side pointing in the $(w_j^n,1)$ direction, and the longest one in the $(w_j^n,-1)$ direction; as well as $\chi_{B_1}((T^{\ell_k^n}_{w_k^n})^{-1}r_k^n(x-x_k^n,t-t_k^n))$ is supported in a parallelepiped $P_2 := \frac{(\ell_k^n)^2}{r_k^n} \times \frac{1}{r_k^n} \times \frac{\ell_k^n}{r_k^n} \times \cdots \times \frac{\ell_k^n}{r_k^n}$ with the smallest side pointing in the $(w_k^n,1)$ direction, and the longest one in the $(w_k^n,-1)$ direction. We have then

$$|P_1 \cap P_2| \lesssim \frac{1}{(r_k^n)^{d+1}} \frac{(\ell_k^n)^d}{|w_k^n - w_j^n|}.$$

Therefore, from (2.45) we get

$$I_n \lesssim \frac{1}{\ell_k^n |w_k^n - w_j^n|}.$$

By (2.5) we deduce the result.

Case 4: The sequences satisfy the Space-time translation property.

Suppose (2.6) holds, then we infer that

$$\text{supp } \chi_{B_1}\left(T^{\ell_j^n}_{w_j^n}(T^{\ell_k^n}_{w_k^n})^{-1}\frac{r_k^n}{r_j^n}(x,t)\right) \subset_{n\to\infty} K, \tag{2.48}$$

with K a fixed compact set.

By (2.6) and (2.48) we deduce that

$$\chi_{B_1}(x,t)\chi_{B_1}\left((T^{\ell_k^n}_{w_k^n})^{-1}T^{\ell_j^n}_{w_j^n}\frac{r_k^n}{r_j^n}(x,t) + (T^{\ell_k^n}_{w_k^n})^{-1}r_k^n(x_j^n - x_k^n, t_j^n - t_k^n)\right) \underset{n\to\infty}{\longrightarrow} 0$$

for all $(x,t) \in \mathbb{R}^{d+1}$, and therefore by (2.46) we are done.

$\square$

Proof of Lemma 2.2. We have to prove that

$$\lim_{n\to\infty} \int g(x,t)(\Gamma_2^n)^{-1}\Gamma_1^n S[\phi_0^1,\phi_1^1](x,t)dxdt = 0$$

where $g \in L^{2\frac{d+1}{d+3}}(\mathbb{R}^{d+1})$. By a change of variables, it is equivalent to prove

$$\lim_{n\to\infty} \int \Gamma_2^n g(x,t)\Gamma_1^n S[\phi_0^1, \phi_1^1](x,t)dxdt = 0,$$

which can be deduced by arguing as in the previous proof.

$\square$

CHAPTER 3

Norm concentration for the nonlinear wave equation

3.1. Introduction

We consider the initial value problem for the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$-critical nonlinear wave equation:

$$\begin{cases} u_{tt} - \Delta u + \gamma u|u|^{\frac{4}{d-1}} = 0 \\ u(0) = u_0 \in \dot{H}^{\frac{1}{2}},\ u_t(0) = u_1 \in \dot{H}^{-\frac{1}{2}}. \end{cases} \tag{3.1}$$

where $\gamma \in \mathbb{R} \setminus \{0\}$ and $d \geq 3$. A function $u : \mathbb{R}^d \times I \to \mathbb{R}$ on an open time interval $I \subset \mathbb{R}$ containing the origin is a solution to (3.1) if $(u, u_t) \in C_t^0(\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}; K)$ and $u \in L_{t,x}^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times K)$ for all compact $K \subset I$, and obeys the Duhamel formula

$$u(t) = S[u_0, u_1](t) + \gamma \int_0^t \frac{1}{2i}\Big(\frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}} - \frac{e^{-i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}\Big) u|u|^{\frac{4}{d-1}}(\tau)d\tau. \tag{3.2}$$

for all $t \in I$.

A solution is said to be *global* if $I = \mathbb{R}$ and a solution is said to blow up if

$$\|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times I)} = \infty.$$

If the norm were finite, the solution could be extended beyond I by standard arguments.

The maximal time interval of existence will be denoted $(T_{\min}, T_{\max})$. In the recent literature *blow-up solutions of type II* have been considered, i.e solutions which blow up and remain bounded in the initial norm; in our case that is

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|^2_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} < \infty.$$

In the case of the defocusing $\dot{H}^1 \times L^2$ energy supercritical wave equation, recent results of Kenig and Merle [43], and Killip and Visan [53], [54], prohibit blow-up solutions of type II; if u is a *blow-up solution*, then

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|^2_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} = \infty.$$

On the other hand, Krieger, Schlag and Tataru [55] constructed type II blow-up solutions for the focusing energy critical wave equation in dimension $d = 3$. Also the work of Duyckaerts, Kenig and Merle [24], [25] characterize these solutions.

We prove that for blow-up solutions of type II there is a concentration of the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ norm. This may help to prohibit the existence of such solutions.

The sets where the solution will be concentrated in space-time, are rectangles in $\mathbb{R}^d$ of dimensions $2^{-k} \times 2^{-k}2^j \times \cdots \times 2^{-k}2^j$, with $k \in \mathbb{R}$, $j \in \mathbb{R}^+$, that we denote by $R^{j,k}$.

Theorem 3.1. *Suppose that u is a solution of (3.1) that blows up at $T_{max} < \infty$. Suppose also that*

$$\sup_{t\in[0,T_{\max})} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t u(t)\|_{\dot{H}^{-\frac{1}{2}}} \leq B. \tag{3.3}$$

Then

$$\limsup_{t\to T_{\max}} \sup_{\substack{R^{j,k},\tau_m^{j',k'}:\\ T_{\max}-t\geq\max(2^{-k}2^{2j},2^{-k'}2^{2j'})}} \|P_{\tau_m^{j',k'}}(\chi_{R^{j,k}}u(t))\|_{\dot{H}^{\frac{1}{2}}} + \|P_{\tau_m^{j',k'}}(\chi_{R^{j,k}}\partial_t u(t))\|_{\dot{H}^{-\frac{1}{2}}} > \epsilon, \tag{3.4}$$

where ϵ depends only on B and γ.

For the L^2-critical Schrödinger equation Bourgain [11] proved a similar result in $\mathbb{R}^{2+1}$ and it was generalized to higher dimensions by Begout and Vargas [3]. See also Rogers and Vargas [67] for the nonelliptic Schrödinger equation, Chae, Hong and Lee [16] for higher order Schrödinger equations, and Chae, Hong, Kim, Lee and Yang [17] for the Hartree equation. In these cases a hypothesis like (3.3) is not needed as the L^2-norm is conserved.

In the following section we present adaptations of lemmas originally due to Bourgain for the Schrödinger equation.

In the third section, we proof the theorem. The main difficulties are generated by the need to control the Fourier supports and the space-time supports simultaneously.

3.2. Preliminary lemmas

For the proof of Theorem 3.1, which is based on an argument of [11], we need some preliminary lemmas. The first is from the second chapter and we write it again for the convenience of the reader.

Lemma 3.1. [64] *Let $(u_0, u_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ with $d \geq 2$ and $\|S[u_0, u_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq \epsilon$. Then, for every $\epsilon > 0$, there exist $N = N(\epsilon, \|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}})$, $A = A(\epsilon, \|(u_0, u_1)\|_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}})$, a family of pairs of functions $\{(f_0^i, f_1^i)\}_{1\leq i\leq N}$ and a family of sectors $\{\tau_{m_i}^{j_i,k_i}\}_{1\leq i\leq N}$ that satisfy*

(i) *compact Fourier support:*

$$\operatorname{supp}(\widehat{f_0^i}),\ \operatorname{supp}(\widehat{f_1^i})\ \subset \tau_{m_i}^{j_i,k_i},$$

(ii) *boundedness:*

$$2^{\frac{k_i}{2}}|\widehat{f_0^i}|, 2^{\frac{-k_i}{2}}|\widehat{f_1^i}| \le A|\tau_{m_i}^{j_i,k_i}|^{-\frac{1}{2}},$$

(iii) *closeness:*

$$\Big\|S[u_0,u_1]-\sum_{i=1}^N S[f_0^i,f_1^i]\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} < \epsilon,$$

(iv) *orthogonality:*

$$\|(u_0,u_1)\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} = \sum_{i=1}^N \|(f_0^i,f_1^i)\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}} + \Big\|(u_0-\sum_{i=1}^N f_0^i, u_1-\sum_{i=1}^N f_1^i)\Big\|^2_{\dot H^{\frac12}\times \dot H^{-\frac12}}.$$

The next lemma takes advantage of the frequency localization to obtain concentration in space-time of $S[f_0^i,f_1^i]$.

Lemma 3.2. *Let $\widehat{f}_0, \widehat{f}_1$ be supported in a sector $\tau_m^{j,k}$ satisfying $2^{\frac{k}{2}}\big|\widehat{f}_0\big|, 2^{-\frac{k}{2}}\big|\widehat{f}_1\big| \le A|\tau_m^{j,k}|^{-\frac12}$. Then, for all $\epsilon>0$, there exist $N=N(\epsilon,A)$, regions $\big\{(\Upsilon_m^{j,k})_i\big\}_{1\le i\le N}$, where $(\Upsilon_m^{j,k})_i$ are parallelepipeds of dimensions $2^{-k}\times 2^{-k}2^{2j}\times 2^{-k}2^j\times\cdots\times 2^{-k}2^j$, with longest side pointing in the direction $(w_m,-1)$ and shortest side pointing in the direction $(w_m,1)$, such that*

$$\|S[f_0,f_1]\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1}\setminus\bigcup(\Upsilon_m^{j,k})_i)} < \epsilon.$$

Proof By changing variables

$$\begin{aligned}
e^{it\sqrt{-\Delta}}f_0(x) &= \frac{2^{kd}}{(2\pi)^d}\int e^{i(2^kx\cdot\xi+2^kt|\xi|)}\widehat{f}_0(2^k\xi)d\xi\\
&= \frac{2^{kd}}{(2\pi)^d}\int |J(T_{0,w_m}^{2^j})^{-1}(\xi)|e^{i\langle (T_{w_m}^{2^j})^{-1}(2^kx,2^kt),(\xi,|\xi|)\rangle}\widehat{f}_0\big((T_{0,w_m}^{2^j})^{-1}(2^k\xi)\big)d\xi\\
&= \frac{2^{\frac{k(d-1)}{2}}2^{\frac{-j(d-1)}{2}}}{(2\pi)^d}\int |J(T_{0,w_m}^{2^j})^{-1}(\xi)|\, e^{i\langle (T_{w_m}^{2^j})^{-1}(2^kx,2^kt),(\xi,|\xi|)\rangle}\\
&\qquad\qquad \times 2^{\frac{k(d+1)}{2}}2^{\frac{j(d-1)}{2}}\widehat{f}_0\big((T_{0,w_m}^{2^j})^{-1}(2^k\xi)\big)d\xi\\
&= 2^{\frac{k(d-1)}{2}}2^{\frac{-j(d-1)}{2}}\ e^{it'\sqrt{-\Delta}}f_0'(x'),
\end{aligned}$$

where $(x',t')=(T_{w_m}^{2^j})^{-1}(2^kx,2^kt)$,

$$\widehat{f}_0'(\xi) = 2^{\frac{k(d+1)}{2}}2^{\frac{j(d-1)}{2}}|J(T_{0,w_m}^{2^j})^{-1}(\xi)|\widehat{f}_0\big((T_{0,w_m}^{2^j})^{-1}(2^k\xi)\big),$$

$(T^{2^j}_{0,w_m})^{-1}$ is the transformation defined as $(T^{2^j}_{w_m})^{-1}(\xi,|\xi|) = \big((T^{2^j}_{0,w_m})^{-1}(\xi), \big|(T^{2^j}_{0,w_m})^{-1}(\xi)\big|\big)$, and $|J(T^{2^j}_{0,w_m})^{-1}(\xi)|$ is the Jacobian of the transformation $(T^{2^j}_{0,w_m})^{-1}$. It is easy to see that $|J(T^{2^j}_{0,w_m})^{-1}(\xi)| \sim 2^{-j(d-1)}$ on the support of $\widehat{f}_0(2^k\xi)$. Thus, $\widehat{f}'_0(\xi)$ is a function supported in the annulus $\mathcal{A}_1$, and satisfies

$$\begin{aligned}|\widehat{f}'_0(\xi)| &\sim 2^{\frac{k(d+1)}{2}} 2^{\frac{-j(d-1)}{2}} \big|\widehat{f}_0\big((T^{2^j}_{0,w_m})^{-1}(2^k\xi)\big)\big| \\ &\le 2^{\frac{k(d+1)}{2}} 2^{\frac{-j(d-1)}{2}} A 2^{-\frac{k}{2}} |\tau^{j,k}_k|^{-\frac{1}{2}} \\ &\le A.\end{aligned}$$

Similarly, we have

$$\frac{e^{it\sqrt{-\Delta}} f_1}{\sqrt{-\Delta}}(x) = 2^{\frac{k(d-1)}{2}} 2^{\frac{-j(d-1)}{2}} \frac{e^{it'\sqrt{-\Delta}} f'_1}{\sqrt{-\Delta}}(x'),$$

where $\widehat{f}'_1(\xi) = 2^{\frac{k(d-1)}{2}} 2^{\frac{j(d-1)}{2}} |J(T^{2^j}_{0,w_m})^{-1}(\xi)| \widehat{f}_1\big((T^{2^j}_{0,w_m})^{-1}(2^k\xi)\big)$, $\widehat{f}'_1$ is supported in the annulus $\mathcal{A}_1$ and $|\widehat{f}'_1(\xi)| \le A$.

Using Wolff's linear restriction theorem [84], there is a $q < 2\frac{d+1}{d-1}$ such that

$$\|e^{it'\sqrt{-\Delta}} f'_0\|_{L^q(\mathbb{R}^{d+1})}, \Big\|\frac{e^{it'\sqrt{-\Delta}} f'_1}{\sqrt{-\Delta}}\Big\|_{L^q(\mathbb{R}^{d+1})} \lesssim (\|\widehat{f}'_0\|_\infty + \|\widehat{f}'_1\|_\infty) \lesssim A,$$

and then, writing $\lambda = \big(\big(\frac{\epsilon}{2}\big)^{2\frac{d+1}{d-1}} A^{-q}\big)^{\frac{1}{2\frac{d+1}{d-1}-q}}$, we get

$$\int_{|e^{it'\sqrt{-\Delta}} f'_0(x')|<\lambda} |e^{it'\sqrt{-\Delta}} f'_0(x')|^{2\frac{d+1}{d-1}} dx'dt' \; < A^q \lambda^{2\frac{d+1}{d-1}-q} \le \Big(\frac{\epsilon}{2}\Big)^{2\frac{d+1}{d-1}}. \tag{3.5}$$

We cover $\Big\{(x',t') : |e^{it'\sqrt{-\Delta}} f'_0(x')| > \lambda\Big\}$ by N_1 balls B_n of radius $\frac{\lambda}{4CA}$, so that

$$\int_{\mathbb{R}^{d+1}\setminus\bigcup B_n} |e^{it\sqrt{-\Delta}} f'_0(x',t')|^{2\frac{d+1}{d-1}} dx'dt' < \Big(\frac{\epsilon}{2}\Big)^{2\frac{d+1}{d-1}}.$$

We are required to prove that N_1 depends only on ϵ and A. To see this we note that if $|e^{it'\sqrt{-\Delta}} f'_0(x')| > \lambda$, $|x''-x'| \le \frac{\lambda}{4AC}$ and $|t''-t'| \le \frac{\lambda}{4AC}$, then, as f'_0 is Fourier compactly supported and bounded by A, we have

$$|e^{it'\sqrt{-\Delta}} f'_0(x') - e^{it''\sqrt{-\Delta}} f'_0(x'')| \le CA(|x'-x''| + |t'-t''|),$$

and we conclude that $|e^{it''\sqrt{-\Delta}} f'_0(x'')| \ge \frac{\lambda}{2}$. Therefore, taking

$$N_1 = \frac{\Big|\Big\{(x',t') : |e^{it'\sqrt{-\Delta}} f'_0(x')| > \frac{\lambda}{2}\Big\}\Big|}{\big(\frac{\lambda}{2AC}\big)^{d+1}},$$

it is enough to take N_1 balls B_n of radius $\frac{\lambda}{2AC}$ to cover $\Big\{(x',t') : |e^{it'\sqrt{-\Delta}} f_0'(x')| > \lambda\Big\}$. Now, by Chebychev and the Strichartz inequality (1.1),

$$\begin{aligned}
\frac{\Big|\Big\{(x',t') : |e^{it'\sqrt{-\Delta}} f_0'(x')| > \frac{\lambda}{2}\Big\}\Big|}{\left(\frac{\lambda}{2AC}\right)^{d+1}} &\leq \left(\frac{2}{\lambda}\right)^{2\frac{d+1}{d-1}} \frac{\|e^{it'\sqrt{-\Delta}} f_0'(x')\|_{2\frac{d+1}{d-1}}^{2\frac{d+1}{d-1}}}{\left(\frac{\lambda}{2AC}\right)^{d+1}} \\
&\leq \left(\frac{2}{\lambda}\right)^{2\frac{d+1}{d-1}} \frac{\|f_0'\|_{\dot{H}^{\frac{1}{2}}}^{2\frac{d+1}{d-1}}}{\left(\frac{\lambda}{2AC}\right)^{d+1}} \\
&\lesssim \left(\frac{2}{\lambda}\right)^{2\frac{d+1}{d-1}} \frac{A^{2\frac{d+1}{d-1}}}{\left(\frac{\lambda}{2AC}\right)^{d+1}},
\end{aligned}$$

and therefore N_1 is bounded by something which depends only on $\lambda(\epsilon, A)$.

Now, defining the parallelepipeds

$$(\Upsilon_m^{j,k})_n' := 2^{-k} T_{w_m}^{2^j}(B_n),$$

we can cover each by a collection of parallelepipeds

$$(\Upsilon_m^{j,k})_n' \subset \bigcup_{n'}^{N_2} (\Upsilon_m^{j,k})_{n,n'}$$

of dimensions $2^{-k} \times 2^{-k}2^{2j} \times 2^{-k}2^j \times \cdots \times 2^{-k}2^j$, with longest side pointing in the direction $(w_m, -1)$ and shortest side pointing in the direction $(w_m, 1)$. The cardinal N_2 of this collection depends only on ϵ and A.

We reorder the collection

$$\bigcup_i^N (\Upsilon_m^{j,k})_i = \bigcup_n^{N_1} \bigcup_{n'}^{N_2} (\Upsilon_m^{j,k})_{n,n'},$$

where N depends only on ϵ and A.

We have

$$
\begin{aligned}
\int_{\mathbb{R}^{d+1}\setminus\bigcup_i(\Upsilon_m^{j,k})_i} & |e^{it\sqrt{-\Delta}}f_0|^{2\frac{d+1}{d-1}}dxdt \\
&\leq \int_{\mathbb{R}^{d+1}\setminus\bigcup_n(\Upsilon_m^{j,k})'_n} |e^{it\sqrt{-\Delta}}f_0|^{2\frac{d+1}{d-1}}dxdt \\
&= 2^{k(d+1)}2^{-j(d+1)} \\
&\quad \times \int_{\mathbb{R}^{d+1}\setminus\bigcup_n(\Upsilon_m^{j,k})'_n} \Big|\int e^{2\pi i\langle (T_{w_m}^{2^j})^{-1}(2^k x,2^k t),(\xi,|\xi|)\rangle}\widehat{f_0'}(\xi)d\xi\Big|^{2\frac{d+1}{d-1}}dxdt \\
&= \int_{\mathbb{R}^{d+1}\setminus\bigcup_n B_n}\Big|\int e^{2\pi i\langle (x',t'),(\xi,|\xi|)\rangle}\widehat{f_0'}(\xi)d\xi\Big|^{2\frac{d+1}{d-1}}dx'dt' \\
&< \Big(\frac{\epsilon}{2}\Big)^{2\frac{d+1}{d-1}}.
\end{aligned}
$$

Similarly, for another collection $\{(\Upsilon_m^{j,k})_i\}_{1\leq i\leq N}$, we obtain

$$
\int_{\mathbb{R}^{d+1}\setminus\bigcup_i(\Upsilon_m^{j,k})_i}\Big|\frac{e^{it\sqrt{-\Delta}}f_1}{\sqrt{-\Delta}}\Big|^{2\frac{d+1}{d-1}}dxdt < \Big(\frac{\epsilon}{2}\Big)^{2\frac{d+1}{d-1}}
$$

and the result holds by taking the union of both collections of parallelepipeds.

□

Remark 3.1. *As* $\angle\big((w_m,-1),(0,\cdots,0,1)\big)=\frac{\pi}{4}$, *we have that*

$$
(\Upsilon_m^{j,k})_i^{t_0} := (\Upsilon_m^{j,k})_i\cap\big\{(x,t)\in\mathbb{R}^{d+1}:\ t=t_0\big\}
$$

is, up to a mild dilation, a rectangle of dimensions $2^{-k}\times 2^{-k}2^j\times\cdots\times 2^{-k}2^j$.

For the convenience of the reader we include the proof of the following well known lemma, which follows by well-known arguments.

Lemma 3.3. *If* u *is a solution of* (3.1) *and* $(T_0,T_1)\subset\mathbb{R}$, *then*

$$
\|u-S[u(T_0),\partial_t u(T_0)](t-T_0)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))} \lesssim |\gamma|\|u\|^{\frac{d+3}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))} \tag{3.6}
$$

Proof. By the Duhamel formula,

$$
u(t)-S[u(T_0),\partial_t u(T_0)](t-T_0)=\gamma\int_{T_0}^t\frac{1}{i}\Big(\frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}-\frac{e^{-i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}\Big)u|u|^{\frac{4}{d-1}}(\tau)d\tau,
$$

so that

$$\begin{aligned}
&\|u(t) - S[u(T_0)\ ,\partial_t u(T_0)](t-T_0)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\\
&\qquad = |\gamma|\Big\|\int_{T_0}^t \frac{1}{2i}\Big(\frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}} - \frac{e^{-i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}\Big)u|u|^{\frac{4}{d-1}}(\tau)d\tau\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\\
&\qquad \le |\gamma|\Big\|\int_{T_0}^t \frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}u|u|^{\frac{4}{d-1}}(\tau)d\tau\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\\
&\qquad + |\gamma|\Big\|\int_{T_0}^t \frac{e^{-i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}u|u|^{\frac{4}{d-1}}(\tau)d\tau\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}.
\end{aligned}$$

So it will suffice to prove

$$\Big\|\int_{T_0}^t \frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}u|u|^{\frac{4}{d-1}}(\tau)d\tau\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))} \le C\|u\|^{\frac{d+3}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}, \tag{3.7}$$

the argument for the second term being the same.

Now, using the Littlewood–Paley inequality,

$$\begin{aligned}
&\Big\|\int_{T_0}^t \frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}u|u|^{\frac{4}{d-1}}(\tau)d\tau\Big\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\\
&\qquad\qquad \lesssim \sum_k \Big\|\int_{T_0}^t \frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}P_k u|u|^{\frac{4}{d-1}}(\tau)d\tau\Big\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))},
\end{aligned}$$

which, by Minkowski integral inequality, is bounded by

$$\sum_k \Big\|\int_{T_0}^t \Big\|\frac{e^{i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}}P_k u|u|^{\frac{4}{d-1}}(\tau)\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d)}d\tau\Big\|^2_{L^{2\frac{d+1}{d-1}}(T_0,T_1)}. \tag{3.8}$$

Now, by Plancherel's theorem, if $\operatorname{supp}\widehat{f_k} \subset \mathcal{A}_k = \{\xi \in \mathbb{R}^d : |\xi| \sim 2^k\}$, then,

$$\Big\|\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f_k\Big\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-k}\|f_k\|_{L^2(\mathbb{R}^d)}.$$

On the other hand, using the well-known decay estimate

$$\|e^{it\sqrt{-\Delta}}f\|_\infty \le |t|^{-\frac{d-1}{2}}\|f\|_1$$

which is valid for functions f such that $\operatorname{supp}\widehat{f} \subset \mathcal{A}_1$,

$$\begin{aligned}
\Big\|\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f_k\Big\|_{L^\infty(\mathbb{R}^d)} &= \Big\|\int e^{i(x\cdot\xi+t|\xi|)}\beta\Big(\frac{\xi}{2^k}\Big)\frac{\widehat{f_k}(\xi)}{|\xi|}d\xi\Big\|_{L^\infty(\mathbb{R}^d)}\\
&\lesssim 2^{k(d-1)}\Big\|\int e^{i(2^k x\cdot\xi+2^k t|\xi|)}\frac{\beta(\xi)}{|\xi|}d\xi\Big\|_{L^\infty(\mathbb{R}^d)}\|f_k\|_{L^1(\mathbb{R}^d)}\\
&\lesssim 2^{k(d-1)}(2^k|t|)^{-\frac{d-1}{2}}\Big\|\Big(\frac{\beta}{|\xi|}\Big)^\vee\Big\|_{L^1(\mathbb{R}^d)}\|f_k\|_{L^1(\mathbb{R}^d)}\\
&\lesssim 2^{\frac{k(d-1)}{2}}|t|^{-\frac{d-1}{2}}\|f_k\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

where β a smooth function adapted to $\mathcal{A}_1$.

Interpolating between the two we obtain

$$\Big\|\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f_k\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d)} \lesssim |t|^{-\frac{d-1}{d+1}}\|f_k\|_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^d)},$$

so that, (3.8) is bounded by

$$\sum_k \Big\|\int_{T_0}^t \frac{\|P_k u|u|^{\frac{4}{d-1}}(\tau)\|_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^d)}}{|t-\tau|^{\frac{d-1}{d+1}}}d\tau\Big\|^2_{L^{2\frac{d+1}{d-1}}(T_0,T_1)}.$$

By the 1-dimensional Hardy–Littlewood–Sobolev inequality

$$\sum_k \Big\|\int_{T_0}^t \frac{\|P_k u|u|^{\frac{4}{d-1}}(\tau)\|_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^d)}}{|t-\tau|^{\frac{d-1}{d+1}}}d\tau\Big\|^2_{L^{2\frac{d+1}{d-1}}(T_0,T_1)} \lesssim \sum_k \|P_k u|u|^{\frac{4}{d-1}}\|^2_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^d\times(T_0,T_1))}.$$

As $\frac{2(d+1)}{d+3} \le 2$, we can exchange the order of the sum and the integral, and apply the Littlewood-Paley inequality, so that

$$\begin{aligned}\sum_k \|P_k u|u|^{\frac{4}{d-1}}\|^2_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^d\times(T_0,T_1))} &\lesssim \|u|u|^{\frac{4}{d-1}}\|^2_{L^{2\frac{d+1}{d+3}}(\mathbb{R}^d\times(T_0,T_1))}\\ &= \|u\|^{2\frac{d+3}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))},\end{aligned}$$

which yields (3.7) and so we are done.

□

3.3. Proof of Theorem 3.1

Concentration in space-time. By hypothesis we have that

$$\|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(0,T_{\max}))} = \infty.$$

Thus, for a small constant η to be determined later, and for every $T_0 < T_{\max}$, there is a $T_1 < T_{\max}$ such that

$$\|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))} = \eta.$$

By Lemma 3.3,

$$\|u - S[u(T_0),\partial_t u(T_0)](t-T_0)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))} \lesssim |\gamma|\eta^{\frac{d+3}{d-1}}, \tag{3.9}$$

so that

$$\|S[u(T_0),\partial_t u(T_0)](t-T_0)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))} \ge \eta - C|\gamma|\eta^{\frac{d+3}{d-1}} > \eta^{\frac{d+3}{d-1}}.$$

By Lemma 3.1, there exist pairs of functions $\{(f_0^i, f_1^i)\}_{1\le i\le N_0}$, Fourier supported in rectangles $\tau_{m_i}^{j_i,k_i}$, such that

$$\Big\|S[u(T_0),\partial_t u(T_0)](t-T_0)-\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)\Big\|_{L^{2\frac{d+1}{d-1}}} \le \eta^{\frac{d+3}{d-1}}, \tag{3.10}$$

$$2^{\frac{k_i}{2}}|\widehat{f_0^i}|, 2^{\frac{-k_i}{2}}|\widehat{f_1^i}| \le A|\tau_{m_i}^{j_i,k_i}|^{-\frac{1}{2}},$$

where N_0 and A depend only on B and η. Now, by Hölder's inequality, (3.9) and (3.10),

$$\begin{aligned}
&\int_{T_0}^{T_1}\int_{\mathbb{R}^d} |u|^2|u-\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)|^{\frac{4}{d-1}}\\
&\quad\le \|u\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\Big\|u-\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)\Big\|^{\frac{4}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\\
&\quad\le \eta^2\Big(\|u-S[u(T_0),\partial_t u(T_0)](t-T_0)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\\
&\qquad\qquad + \Big\|S[u(T_0),\partial_t u(T_0)](t-T_0)-\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)\Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\Big)^{\frac{4}{d-1}}\\
&\quad\le \eta^2(C|\gamma|\eta^{\frac{d+3}{d-1}}+\eta^{\frac{d+3}{d-1}})^{\frac{4}{d-1}}\\
&\quad\le \frac{1}{4}\eta^{2\frac{d+1}{d-1}},
\end{aligned}$$

where we choose η small enough to satisfy $\eta^{\frac{4}{d-1}} \le 4^{-\frac{d-1}{4}}(C|\gamma|+1)^{-1}$.

For every $a, b \ge 0$,

$$(a+b)^\alpha \le C(\alpha)(a^\alpha+b^\alpha), \quad \text{with } \begin{cases} C(\alpha)=1 & \text{if } 0<\alpha\le 1\\ C(\alpha)=2^{\alpha-1} & \text{if } \alpha>1\end{cases}, \tag{3.11}$$

and in particular for $d \ge 3$, we have $(a+b)^{\frac{4}{d-1}} \le 2(a^{\frac{4}{d-1}}+b^{\frac{4}{d-1}})$, so that

$$\begin{aligned}
\eta^{2\frac{d+1}{d-1}} = \int_{T_0}^{T_1}\int_{\mathbb{R}^d}|u|^2|u|^{\frac{4}{d-1}} \le\ & 2\int_{T_0}^{T_1}\int_{\mathbb{R}^d}|u|^2|\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)|^{\frac{4}{d-1}}\\
&+2\int_{T_0}^{T_1}\int_{\mathbb{R}^d}|u|^2|u-\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)|^{\frac{4}{d-1}}.
\end{aligned}$$

This yields

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^d}|u|^2|\sum_{i=0}^{N_0} S[f_0^i,f_1^i](t-T_0)|^{\frac{4}{d-1}} \ge \frac{1}{4}\eta^{2\frac{d+1}{d-1}}.$$

That is, there exists an i_0 such that

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^d}|u|^2|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}}\geq\epsilon_0, \tag{3.12}$$

where ϵ_0 depends only on $\eta(\gamma)$ and B.

Now, by Lemma 3.2, and setting $j=j_{i_0},k=k_{i_0},m=m_{i_0}$, we can find a collection $\{(\Upsilon_m^{j,k})_i\}_{1\leq i\leq N_1}$ for which

$$\int_{\mathbb{R}^{d+1}\setminus\bigcup(\Upsilon_m^{j,k})_i}|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{2\frac{d+1}{d-1}}\leq\left(\frac{\epsilon_0}{2\eta^2}\right)^{\frac{d+1}{2}},$$

where N_1 depends on B and γ. By Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d\times(T_0,T_1)\setminus\bigcup(\Upsilon_m^{j,k})_i}&|u|^2|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}}\\&\leq\|u\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\left(\int_{\mathbb{R}^{d+1}\setminus\bigcup(\Upsilon_m^{j,k})_i}|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{2\frac{d+1}{d-1}}\right)^{\frac{2}{d+1}}\leq\frac{\epsilon_0}{2}.\end{aligned}$$

Thus, by (3.12) we have that

$$\int_{\mathbb{R}^d\times(T_0,T_1)\cap\bigcup(\Upsilon_m^{j,k})_i}|u|^2|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}}\geq\frac{\epsilon_0}{2},$$

and we can find a $(\Upsilon_m^{j,k})_{i_0}$ such that

$$\int_{\mathbb{R}^d\times(T_0,T_1)\cap(\Upsilon_m^{j,k})_{i_0}}|u|^2|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}}\geq\frac{\epsilon_0}{2N_1}=\epsilon_1.$$

We rewrite this as (see Remark 3.1),

$$\int_{(T_0,T_1)\cap I_0}\int_{(\Upsilon_m^{j,k})_{i_0}^t}|u|^2|S[f_0^{i_0},f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}}\geq\epsilon_1, \tag{3.13}$$

where $(\Upsilon_m^{j,k})_{i_0}^t$ is a mild dilation of a rectangle in $\mathbb{R}^d$ of dimensions $2^{-k}\times2^{-k}2^j\times\cdots\times2^{-k}2^j$ and $|I_0|\sim2^{-k}2^{2j}$. Now

$$\begin{aligned}|S[f_0^{i_0},f_1^{i_0}](t-T_0)|&\leq\frac{1}{2}\int_{\tau_m^{j,k}}\left|\widehat{f_0^{i_0}}(\xi)\right|d\xi+\frac{1}{2}\int_{\tau_m^{j,k}}\frac{\left|\widehat{f_1^{i_0}}(\xi)\right|}{|\xi|}d\xi\\&\leq\frac{1}{2}|\tau_m^{j,k}|2^{-\frac{k}{2}}A|\tau_m^{j,k}|^{-\frac{1}{2}}+\frac{1}{2}2^{-k}|\tau_m^{j,k}|2^{\frac{k}{2}}A|\tau_m^{j,k}|^{-\frac{1}{2}}=2^{-\frac{k}{2}}A|\tau_m^{j,k}|^{\frac{1}{2}},\end{aligned} \tag{3.14}$$

so that by Hölder's inequality,

$$\begin{aligned}\epsilon_1 &\leq \int_{T_0}^{T_1}\int_{(\Upsilon_m^{j,k})_{i_0}^t} |u|^2 |S[f_0^{i_0}, f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}} \\ &\leq \|u\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(T_0,T_1))}\Big(\int_{T_0}^{T_1}\int_{(\Upsilon_m^{j,k})_{i_0}^t} |S[f_0^{i_0}, f_1^{i_0}](t-T_0)|^{2\frac{d+1}{d-1}}\Big)^{\frac{2}{d+1}} \\ &\leq \eta^2 (T_1-T_0)^{\frac{2}{d+1}} |(\Upsilon_m^{j,k})_{i_0}^t|^{\frac{2}{d+1}} 2^{-\frac{2k}{d-1}} A^{\frac{4}{d-1}} |\tau_m^{j,k}|^{\frac{2}{d-1}} \\ &\lesssim \eta^2 A^{\frac{4}{d-1}} (T_1-T_0)^{\frac{2}{d+1}} 2^{k\frac{2}{d+1}} 2^{-j\frac{4}{d+1}}.\end{aligned} \tag{3.15}$$

Therefore,

$$(T_1 - T_0) \geq \epsilon_1^{\frac{d+1}{2}} A^{-2\frac{d+1}{d-1}} \eta^{-(d+1)} 2^{-k} 2^{2j} = \epsilon_2 2^{-k} 2^{2j}. \tag{3.16}$$

Now, arguing as before in (3.15), we also have

$$\int_{T_1-\frac{1}{2}\epsilon_2 2^{-k}2^{2j}}^{T_1}\int_{(\Upsilon_m^{j,k})_{i_0}^t} |u|^2 |S[f_0^{i_0}, f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}} \leq \frac{\epsilon_1}{2^{\frac{2}{d+1}}}.$$

Thus, calling

$$J = (T_0, T_1 - \frac{1}{2}\epsilon_2 2^{-k} 2^{2j}) \cap I_0, \tag{3.17}$$

by (3.13) we have

$$\int_J\int_{(\Upsilon_m^{j,k})_{i_0}^t} |u|^2 |S[f_0^{i_0}, f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}} \geq \big(1 - \frac{1}{2^{\frac{2}{d+1}}}\big)\epsilon_1.$$

Now, by (3.14) we deduce

$$\begin{aligned}\epsilon_3 = \big(1 - \frac{1}{2^{\frac{2}{d+1}}}\big)\epsilon_1 A^{-\frac{4}{d-1}} &\leq A^{-\frac{4}{d-1}} \int_J\int_{(\Upsilon_m^{j,k})_{i_0}^t} |u|^2 |S[f_0^{i_0}, f_1^{i_0}](t-T_0)|^{\frac{4}{d-1}} \\ &\leq 2^{2k} 2^{-2j} \int_J\int_{(\Upsilon_m^{j,k})_{i_0}^t} |u|^2.\end{aligned} \tag{3.18}$$

Unbounded oscillation. Let C_1 a large constant to be determined later and break

$$(T_0, T_1) = \bigcup_{s=1}^{\eta^{-C_1}} (t_s, t_{s+1}),$$

so that

$$\|u\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(t_s,t_{s+1}))} = \eta^{C_1}\eta^{2\frac{d+1}{d-1}}, \tag{3.19}$$

for every $1 \le s \le \eta^{-C_1}$. We apply Lemma 3.1 to $(u(t_s), \partial_t u(t_s))$ and obtain pairs of functions $\{(f_0^{s,i}, f_1^{s,i})\}_{1 \le i \le N_2}$ Fourier supported in rectangles $\tau_{m'_{s,i}}^{j'_{s,i},k'_{s,i}}$ such that

$$\Big\| S[u(t_s), \partial_t u(t_s)](t - t_s) - \sum_{i=0}^{N_2} S[f_0^{s,i}, f_1^{s,i}](t - t_s) \Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times (t_s, t_{s+1}))} \le \eta^{C_1} \tag{3.20}$$
$$2^{\frac{k'_{s,i}}{2}} |\widehat{f}_0^{s,i}|, 2^{\frac{-k'_{s,i}}{2}} |\widehat{f}_1^{s,i}| \le A |\tau_{m'_{s,i}}^{j'_{s,i},k'_{s,i}}|^{-\frac{1}{2}},$$

where N_2 and A depend only on B, $\eta(\gamma)$ and C_1. Moreover, reasoning as for (3.14), we have

$$|S[f_0^{s,i}, f_1^{s,i}](t - t_s)| \le 2^{-\frac{k'_{s,i}}{2}} A |\tau_{m'_{s,i}}^{j'_{s,i},k'_{s,i}}|^{\frac{1}{2}}. \tag{3.21}$$

We write

$$\begin{aligned}
&\int_J \int_{(\Upsilon_m^{j,k})_{i_0}^t} |u|^2 = \sum_s \int_{(t_s,t_{s+1}) \cap J} \int_{(\Upsilon_m^{j,k})_{i_0}^t} u\, u \\
&= \sum_s \int_{(t_s,t_{s+1}) \cap J} \int_{(\Upsilon_m^{j,k})_{i_0}^t} u \Big(u - S[u(t_s), \partial_t u(t_s)](t - t_s) \Big) \\
&+ \sum_s \int_{(t_s,t_{s+1}) \cap J} \int_{(\Upsilon_m^{j,k})_{i_0}^t} u \Big(S[u(t_s), \partial_t u(t_s)](t - t_s) - \sum_{i=0}^{N_2} S[f_0^{s,i}, f_1^{s,i}](t - t_s) \Big) \\
&+ \sum_s \int_{(t_s,t_{s+1}) \cap J} \int_{(\Upsilon_m^{j,k})_{i_0}^t} u \Big(\sum_{i=0}^{N_2} S[f_0^{s,i}, f_1^{s,i}](t - t_s) \Big) \\
&= I + II + III.
\end{aligned}$$

First we note that by Hölder's inequality and Lemma 3.3,

$$\begin{aligned}
I &\le \sum_s \Big(|(\Upsilon_m^{j,k})_{i_0}^t| |t_{s+1} - t_s| \Big)^{\frac{2}{d+1}} \\
&\qquad \|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times (t_s, t_{s+1}))} \|u - S[u(t_s), \partial_t u(t_s](t - t_s)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times (t_s, t_{s+1}))} \\
&\le |\gamma| \big|(\Upsilon_m^{j,k})_{i_0}^t\big|^{\frac{2}{d+1}} \sum_s |t_{s+1} - t_s|^{\frac{2}{d+1}} \|u\|^{2\frac{d+1}{d-1}}_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times (t_s, t_{s+1}))},
\end{aligned}$$

so that by (3.19) and Hölder's inequality,

$$\begin{aligned}
I &\le C|\gamma| \big|(\Upsilon_m^{j,k})_{i_0}^t\big|^{\frac{2}{d+1}} \sum_s |t_{s+1} - t_s|^{\frac{2}{d+1}} \eta^{C_1} \eta^{2\frac{d+1}{d-1}} \\
&\le C|\gamma| \big|(\Upsilon_m^{j,k})_{i_0}^t\big|^{\frac{2}{d+1}} \Big(\sum_s |t_{s+1} - t_s| \Big)^{\frac{2}{d+1}} \Big(\sum_s 1 \Big)^{\frac{d-1}{d+1}} \eta^{C_1} \eta^{2\frac{d+1}{d-1}} \\
&\lesssim C|\gamma| \Big(|J| \big|(\Upsilon_m^{j,k})_{i_0}^t\big| \Big)^{\frac{2}{d+1}} \eta^{C_1 \frac{2}{d+1}} \eta^{2\frac{d+1}{d-1}}.
\end{aligned}$$

As we have the following bound on $|J|$,

$$|J| = \left|(T_0, T_1 - \frac{1}{2}\epsilon_2 2^{-k}2^{2j}) \cap I_0\right| \leq \left|I_0\right| \sim 2^{-k}2^{2j},$$

we conclude that

$$I \leq C|\gamma|2^{-2k}2^{2j}\eta^{C_1\frac{2}{d+1}}\eta^{2\frac{d+1}{d-1}}. \tag{3.22}$$

By (3.20) and arguing as before,

$$\begin{aligned}
II &\leq \sum_s \left(|t_{s+1} - t_s|\left|(\Upsilon_m^{j,k})_{i_0}^t\right|\right)^{\frac{2}{d+1}} \|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(t_s,t_{s+1}))} \\
&\qquad \times \|S[u(t_s), \partial_t u(t_s)](t - t_s) - \sum_{i=0}^{N_2} S[f_0^{s,i}, f_1^{s,i}](t - t_s)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d\times(t_s,t_{s+1}))} \\
&\leq \left(|J|\left|(\Upsilon_m^{j,k})_{i_0}^t\right|\right)^{\frac{2}{d+1}} \eta^{-C_1\frac{d-1}{d+1}}\eta^{C_1\frac{d-1}{2(d+1)}}\eta\eta^{C_1} \\
&\leq 2^{-2k}2^{2j}\eta^{C_1\frac{d+3}{2(d+1)}}\eta.
\end{aligned} \tag{3.23}$$

Finally, by Lemma 3.2, for every pair of functions $(f_0^{s,i}, f_1^{s,i})$, we have a collection of regions $\left\{(\Upsilon_{m'_{s,i}}^{j'_{s,i},k'_{s,i}})_r\right\}_{1\leq r\leq N_3}$ such that

$$\|S[f_0^{s,i}, f_1^{s,i}](t - t_s)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1}\setminus\bigcup_r(\Upsilon_{m'_{s,i}}^{j'_{s,i},k'_{s,i}})_r)} \leq \eta^{C_2},$$

where C_2 will be determined later and N_3 depends only on $\eta(\gamma)$, B and C_2. We write

$$\begin{aligned}
(\Upsilon_{m'_{s,i}}^{j'_{s,i},k'_{s,i}})_r \cap (\Upsilon_m^{j,k})_{i_0} &:= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} :\ t \in I_{i_0,s,i,r},\ x \in ((\Upsilon_{m'_{s,i}}^{j'_{s,i},k'_{s,i}})_r \cap (\Upsilon_m^{j,k})_{i_0})^t\} \\
&= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} :\ t \in I_{i_0,s,i,r},\ x \in (\Upsilon_{m'_{s,i}}^{j'_{s,i},k'_{s,i}})_r^t \cap (\Upsilon_m^{j,k})_{i_0}^t\},
\end{aligned}$$

where $(\Upsilon_{m'_{s,i}}^{j'_{s,i},k'_{s,i}})_r^t$, $(\Upsilon_m^{j,k})_{i_0}^t$ are mild dilations of rectangles of dimensions $2^{-k'_{s,i}}\times 2^{-k'_{s,i}}2^{j'_{s,i}}\times \cdots \times 2^{-k'_{s,i}}2^{j'_{s,i}}$ and $2^{-k}\times 2^{-k}2^j \times \cdots \times 2^{-k}2^j$ respectively, and $I_{i_0,s,i,r} \subset J$ is an interval that satisfies

$$|I_{i_0,s,i,r}| \leq \min(2^{2j}2^{-k}, 2^{2j'_{s,i}}2^{-k'_{s,i}}). \tag{3.24}$$

By Hölder's inequality, we obtain

$$\begin{aligned} III \leq & \sum_s \sum_i \sum_r \Big| \int_{I_{i_0,s,i,r} \cap (t_s,t_{s+1})} \int_{(\Upsilon^{j'_{s,i},k'_{s,i}}_{m'_{s,i}})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big| \\ & + \sum_s \Big(|t_{s+1}-t_s| |(\Upsilon^{j,k}_m)^{t_0}_{i_0}| \Big)^{\frac{2}{d+1}} \|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times (t_s,t_{s+1}))} \\ & \qquad \sum_i \|S[f_0^{s,i}, f_1^{s,i}](t-t_s)\|_{L^{2\frac{d+1}{d-1}}\big((\mathbb{R}^{d+1} \setminus \cup_r (\Upsilon^{j'_{s,i},k'_{s,i}}_{m'_{s,i}})_r) \cap (\Upsilon^{j,k}_m)_{i_0}\big)} \\ \leq & \sum_s \sum_i \sum_r \Big| \int_{I_{i_0,s,i,r} \cap (t_s,t_{s+1})} \int_{(\Upsilon^{j'_{s,i},k'_{s,i}}_{m'_{s,i}})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big| \\ & + (|J| |(\Upsilon^{j,k}_m)^{t_0}_{i_0}|)^{\frac{2}{d+1}} \eta^{-C_1 \frac{d-1}{d+1}} \eta^{C_1 \frac{d-1}{2(d+1)}} \eta N_2 \eta^{C_2}. \end{aligned}$$

As N_2 does not depend on C_2, we can choose C_2 so that $\eta^{-C_1 \frac{d-1}{2(d+1)}} \eta N_2 \eta^{C_2} \leq \eta^{C_1}$, and therefore

$$\begin{aligned} III \leq & \sum_s \sum_i \sum_r \Big| \int_{I_{i_0,s,i,r} \cap (t_s,t_{s+1})} \int_{(\Upsilon^{j'_{s,i},k'_{s,i}}_{m'_{s,i}})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big| \\ & + 2^{-2k} 2^{2j} \eta^{C_1}. \end{aligned} \tag{3.25}$$

Thus, by (3.22), (3.23) and (3.25) we have

$$\begin{aligned} & \int_J \int_{(\Upsilon^{j,k}_m)^t_{i_0}} |u|^2 \\ & \qquad \leq \sum_s \sum_i \sum_r \Big| \int_{I_{i_0,s,i,r} \cap (t_s,t_{s+1})} \int_{(\Upsilon^{j'_{s,i},k'_{s,i}}_{m'_{s,i}})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big| \\ & \qquad \quad + 2^{-2k} 2^{2j} \big(C|\gamma| \eta^{C_1 \frac{2}{d+1}} \eta^{2\frac{d+1}{d-1}} + \eta^{C_1 \frac{d+3}{2(d+1)}} \eta + \eta^{C_1} \big). \end{aligned} \tag{3.26}$$

Taking C_1 sufficiently large, we can take

$$\epsilon_4 = \epsilon_3 - \big(C|\gamma| \eta^{C_1 \frac{2}{d+1}} \eta^{2\frac{d+1}{d-1}} + \eta^{C_1 \frac{d+3}{2(d+1)}} \eta + \eta^{C_1} \big) > \frac{\epsilon_3}{2}.$$

Therefore by (3.18),

$$\epsilon_4 \leq 2^{2k} 2^{-2j} \sum_s^{\eta^{-C_1}} \sum_i^{N_2} \sum_r^{N_3} \Big| \int_{I_{i_0,s,i,r} \cap (t_s,t_{s+1})} \int_{(\Upsilon^{j'_{s,i},k'_{s,i}}_{m'_{s,i}})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big|.$$

By the pigeonhole principle, and writing $j' = j'_{s,i}$, $k' = k'_{s,i}$, $m' = m'_{s,i}$, we get

$$\epsilon_5 \leq 2^{2k} 2^{-2j} \Big| \int_{I_{i_0,s,i,r} \cap (t_s,t_{s+1})} \int_{(\Upsilon^{j',k'}_{m'})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big|, \tag{3.27}$$

where ϵ_5 depends only on B and $\eta(\gamma)$.

We can suppose that
$$2^{2k}2^{-2j} \leq \frac{\epsilon_3}{(\epsilon_5)^2}\, 2^{2k'}2^{-2j'}$$
as otherwise we can prove (3.18) with $2^{2k}2^{-2j}$ replaced by $\frac{\epsilon_3}{(\epsilon_5)^2}\, 2^{2k'}2^{-2j'}$ so we could then repeat the argument to obtain (3.27) with $2^{2k}2^{-2j}$ replaced by $\frac{\epsilon_3}{(\epsilon_5)^2}\, 2^{2k'}2^{-2j'}$.

To see this, note that by thw Cauchy-Schwarz inequality
$$\begin{aligned}\epsilon_5 \leq 2^{2k}2^{-2j}\Big(&\int_{I_{i_0,s,i,r}\cap(t_s,t_{s+1})}\int_{(\Upsilon^{j',k'}_{m'})^t_r\cap(\Upsilon^{j,k}_m)^t_{i_0}} |u|^2\Big)^{\frac12}\\ \Big(&\int_{I_{i_0,s,i,r}\cap(t_s,t_{s+1})}\int_{(\Upsilon^{j',k'}_{m'})^t_r\cap(\Upsilon^{j,k}_m)^t_{i_0}} \big|S[f_0^{s,i},f_1^{s,i}](t-t_s)\big|^2\Big)^{\frac12}.\end{aligned}$$
Now, as
$$|(\Upsilon^{j',k'}_{m'})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0}| = \min(2^{-k},2^{-k'})\min(2^{-k(d-1)}2^{j(d-1)},2^{-k'(d-1)}2^{j'(d-1)}), \tag{3.28}$$
together with (3.24) and (3.21), we have that
$$\begin{aligned}\epsilon_5 \leq\, & 2^{2k}2^{-2j}\min(2^j2^{-\frac{k}{2}},2^{j'}2^{-\frac{k'}{2}})\min(2^{-\frac{k}{2}},2^{-\frac{k'}{2}})\min(2^{-k\frac{(d-1)}{2}}2^{j\frac{(d-1)}{2}},2^{-k'\frac{(d-1)}{2}}2^{j'\frac{(d-1)}{2}})\\ &\times 2^{k'\frac{(d-1)}{2}}2^{-j'\frac{(d-1)}{2}}\Big(\int_{I_{i_0,s,i,r}\cap(t_s,t_{s+1})}\int_{(\Upsilon^{j',k'}_{m'})^t_r\cap(\Upsilon^{j,k}_m)^t_{i_0}} |u|^2\Big)^{\frac12}.\end{aligned}$$
Now, we have
$$\begin{aligned}\epsilon_5 &\leq 2^{2k}2^{-2j}2^j2^{-\frac{k}{2}}2^{-\frac{k}{2}}2^{-k\frac{(d-1)}{2}}2^{j\frac{(d-1)}{2}}2^{k'\frac{(d-1)}{2}}2^{-j'\frac{(d-1)}{2}}\\ &\qquad\qquad \times\Big(\int_{I_{i_0,s,i,r}\cap(t_s,t_{s+1})}\int_{(\Upsilon^{j',k'}_{m'})^t_r\cap(\Upsilon^{j,k}_m)^t_{i_0}} |u|^2\Big)^{\frac12}\\ &\leq 2^{k'}2^{-j'}\Big(\int_{I_{i_0,s,i,r}\cap(t_s,t_{s+1})}\int_{(\Upsilon^{j',k'}_{m'})^t_r\cap(\Upsilon^{j,k}_m)^t_{i_0}} |u|^2\Big)^{\frac12},\end{aligned}$$
where we have used that $2^{2k}2^{-2j} > \frac{(\epsilon_5)^2}{\epsilon_3}\, 2^{2k}2^{-2j} > 2^{2k'}2^{-2j'}$ and that $d \geq 3$.

As
$$I_{i_0,s,i,r}\cap(t_s,t_{s+1}) \subset J \quad \text{and} \quad (\Upsilon^{j',k'}_{m'})^t_r \cap (\Upsilon^{j,k}_m)^t_{i_0} \subset (\Upsilon^{j,k}_m)^t_{i_0},$$
this yields
$$\epsilon_3 \leq \frac{\epsilon_3}{(\epsilon_5)^2}\, 2^{2k'}2^{-2j'}\int_J\int_{(\Upsilon^{j,k}_m)^t_{i_0}} |u|^2,$$
and so we have (3.18) with $2^{2k}2^{-2j}$ replaced by $\frac{\epsilon_3}{(\epsilon_5)^2}2^{2k'}2^{-2j'}$.

Therefore,
$$\epsilon_6 \leq 2^{2k'}2^{-2j'}\Big|\int_{I_{i_0,s,i,r}\cap(t_s,t_{s+1})}\int_{(\Upsilon^{j',k'}_{m'})^t_r\cap(\Upsilon^{j,k}_m)^t_{i_0}} u\cdot S[f_0^{s,i},f_1^{s,i}](t-t_s)\Big|, \tag{3.29}$$
where ϵ_6 depends only on $\eta(\gamma)$ and B.

We now write

$$(T_s, T_{s+1}) := I_{i_0,s,i,r} \cap (t_s, t_{s+1}).$$

By Hölder's inequality,(3.29) yields

$$\epsilon_6 \le 2^{2k'} 2^{-2j'} \|u\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^d \times (T_0,T_1))} \|S[f_0^{s,i}, f_1^{s,i}](t-t_s)\|_\infty \times (T_{s+1}-T_s)^{\frac{d+3}{2(d+1)}} \Big|(\Upsilon_{m'}^{j',k'})_r^t \cap (\Upsilon_m^{j,k})_{i_0}^t\Big|^{\frac{d+3}{2(d+1)}}.$$

By (3.21) and (3.28)

$$\epsilon_6 \le 2^{2k'} 2^{-2j'} \eta\, A\, 2^{k'\frac{(d-1)}{2}} 2^{-j'\frac{(d-1)}{2}} (T_{s+1}-T_s)^{\frac{d+3}{2(d+1)}} 2^{-k'\frac{d(d+3)}{2(d+1)}} 2^{j'\frac{(d+3)(d-1)}{2(d+1)}}.$$

Therefore,

$$(T_{s+1}-T_s) \ge \big(\epsilon_6 (\eta\, A\,)^{-1}\big)^{2\frac{d+1}{d+3}} 2^{-k'} 2^{2j'} = \epsilon_7 2^{-k'} 2^{2j'}.$$

Arguing as before, we also have

$$2^{2k'} 2^{-2j'} \Big| \int_{T_{s+1}-\frac{1}{2}\epsilon_7 2^{-k'} 2^{2j'}}^{T_{s+1}} \int_{(\Upsilon_{m'}^{j',k'})_r^t \cap (\Upsilon_m^{j,k})_{i_0}^t} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big| \le \frac{\epsilon_6}{2^{\frac{d+3}{2(d+1)}}}$$

If we write

$$J' = (T_s, T_{s+1} - \frac{1}{2}\epsilon_7 2^{-k'} 2^{2j'}) = (J'_0, J'_1), \tag{3.30}$$

by (3.29) we have

$$\epsilon_8 = (1 - \frac{1}{2^{\frac{d+3}{2(d+1)}}})\epsilon_6 \le 2^{2k'} 2^{-2j'} \Big| \int_{J'} \int_{(\Upsilon_{m'}^{j',k'})_r^t \cap (\Upsilon_m^{j,k})_{i_0}^t} u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big|.$$

We write now for the sake of notational compactness

$$\tau := (\Upsilon_{m'}^{j',k'})_r^t \cap (\Upsilon_m^{j,k})_{i_0}^t, \quad \tau' := \tau_{m'}^{j',k'}.$$

Using the Fourier compact support of $S[f_0^{s,i}, f_1^{s,i}](t-t_s)$,

$$\begin{aligned} \epsilon_8 &\le 2^{2k'} 2^{-2j'} \Big| \int_{J'} \int_\tau u \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big| \\ &= 2^{2k'} 2^{-2j'} \Big| \int_{J'} \int P_{\tau'}\big(\chi_\tau u(t)\big) \cdot S[f_0^{s,i}, f_1^{s,i}](t-t_s) \Big|, \end{aligned}$$

where P_τ is the Fourier multiplier defined as

$$\widehat{P_{\tau'} f} = \chi_{\tau'} \widehat{f}.$$

As we have

$$\begin{aligned} S[f_0^{s,i}, f_1^{s,i}](t) &= \frac{1}{2}\big(e^{it\sqrt{-\Delta}} f_0^{s,i} + e^{-it\sqrt{-\Delta}} f_0^{s,i}\big) + \frac{1}{2i}\Big(\frac{e^{it\sqrt{-\Delta}} f_1^{s,i}}{\sqrt{-\Delta}} - \frac{e^{-it\sqrt{-\Delta}} f_1^{s,i}}{\sqrt{-\Delta}}\Big) \\ &= \frac{1}{2i}\partial_t\Big(\frac{e^{it\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} - \frac{e^{-it\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}}\Big) + \frac{1}{2i}\Big(\frac{e^{it\sqrt{-\Delta}} f_1^{s,i}}{\sqrt{-\Delta}} - \frac{e^{-it\sqrt{-\Delta}} f_1^{s,i}}{\sqrt{-\Delta}}\Big), \end{aligned} \tag{3.31}$$

by an integration by parts and using the fact that $\partial_t(f * g) = f * \partial_t g$, followed by the Cauchy-Schwarz inequality,

$$\begin{aligned}\epsilon_8 \leq 2^{2k'}2^{-2j'} \int_{J'} \Bigg(&\Big(\int |P_{\tau'}\big(\chi_\tau u(t)\big)|^2\Big)^{\frac{1}{2}} \Big(\int \Big| \frac{e^{i(t-t_s)\sqrt{-\Delta}} f_1^{s,i}}{\sqrt{-\Delta}} \Big|^2 \Big)^{\frac{1}{2}} \\ &+ \Big(\int |P_{\tau'}\big(\chi_\tau \partial_t u(t)\big)|^2\Big)^{\frac{1}{2}} \Big(\int \Big| \frac{e^{i(t-t_s)\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} \Big|^2 \Big)^{\frac{1}{2}} \Bigg) \\ + 2^{2k'}2^{-2j'} \Big(\int |P_{\tau'}\big(\chi_\tau u(J_1')\big)|^2\Big)^{\frac{1}{2}} &\Big(\int \Big| \frac{e^{i(J_1'-t_s)\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} - \frac{e^{-i(J_1'-t_s)\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} \Big|^2 \Big)^{\frac{1}{2}} \\ + 2^{2k'}2^{-2j'} \Big(\int |P_{\tau'}\big(\chi_\tau u(J_0')\big)|^2\Big)^{\frac{1}{2}} &\Big(\int \Big| \frac{e^{i(J_0'-t_s)\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} - \frac{e^{-i(J_0'-t_s)\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} \Big|^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

Now by Plancherel's theorem and the Fourier support and boundedness properties of $(f_0^{s,i}, f_1^{s,i})$ as in (3.21) we have

$$2^{k'} \Big(\int \Big| \frac{e^{i(t-t_s)\sqrt{-\Delta}} f_0^{s,i}}{\sqrt{-\Delta}} \Big|^2 \Big)^{\frac{1}{2}}, \quad \Big(\int \Big| \frac{e^{i(t-t_s)\sqrt{-\Delta}} f_1^{s,i}}{\sqrt{-\Delta}} \Big|^2 \Big)^{\frac{1}{2}} \lesssim A 2^{-\frac{k'}{2}},$$

so that, as $j' > 0$, taking the supremum in t, there exists t_0 such that

$$\epsilon_9 \lesssim 2^{2k'}2^{-2j'}|J'| \Bigg(2^{-\frac{k'}{2}} \Big(\int |P_{\tau'}\big(\chi_\tau u(t_0)\big)|^2 \Big)^{\frac{1}{2}} + 2^{-\frac{3k'}{2}} \Big(\int |P_{\tau'}\big(\chi_\tau \partial_t u(t_0)\big)|^2 \Big)^{\frac{1}{2}} \Bigg).$$

where ϵ_9 depends only on $\eta(\gamma)$ and B.

By (3.24), we conclude that

$$\epsilon_{10} \leq \Bigg(2^{\frac{k'}{2}} \Big(\int |P_{\tau'}\big(\chi_\tau u(t_0)\big)|^2 \Big)^{\frac{1}{2}} + 2^{-\frac{k'}{2}} \Big(\int |P_{\tau'}\big(\chi_\tau \partial_t u(t_0)\big)|^2 \Big)^{\frac{1}{2}} \Bigg).$$

where ϵ_{10} depends only on B and $\eta(\gamma)$.

Now, by (3.17) and (3.30), we have

$$T_{\max} - t_0 \geq \frac{\epsilon_2}{2} 2^{-k} 2^{2j} + \frac{\epsilon_7}{2} 2^{-k'} 2^{2j'}.$$

Dividing τ in rectangles of dimensions $\epsilon_2 2^{-k} \times \epsilon_2 2^{-k} 2^j \times \cdots \times \epsilon_2 2^{-k} 2^j$ there will be one which we denote again by τ, and dividing τ' in rectangles of dimensions $\epsilon_7 2^{k'} \times \epsilon_7 2^{k'} 2^{-j'} \times \cdots \times \epsilon_7 2^{k'} 2^{-j'}$ there will be one which we denote again by τ' such that

$$\epsilon(\gamma, B) \leq 2^{k'} \int |P_{\tau'}\big(\chi_\tau u(\cdot, t_0)\big)|^2 + 2^{-k'} \int |P_{\tau'}\big(\chi_\tau \partial_t u(\cdot, t_0)\big)|^2,$$

which completes the proof.

$\square$

Frequently used notation

$A \lesssim B$: $A \leq CB$.

$A \sim B$: $A \lesssim B$ and $A \gtrsim B$.

$\mathcal{A}_k = \{\xi \in \mathbb{R}^d;\ \ 2^k \leq |\xi| \leq 2^{k+1}\}$.

$A_k = \mathcal{A}_{k-1} \cup \mathcal{A}_k \cup \mathcal{A}_{k+1}$.

$\widehat{P_k g} = \chi_{\mathcal{A}_k} \widehat{g}$.

$\|f\|_{\dot{H}^s} = (\sum_k 2^{2ks} \|P_k f\|_2^2)^{\frac{1}{2}}$.

$\|f\|_{\dot{B}^s_{2,q}} = (\sum_k 2^{qks} \|P_k f\|_2^q)^{\frac{1}{q}}$.

$\widehat{\sqrt{-\Delta} f}(\xi) = |\xi| \widehat{f}(\xi)$.

$\tau_m^{j,k} := \left\{ \xi \in \mathcal{A}_k : \left| \frac{\xi}{|\xi|} - w_m \right| \leq \left| \frac{\xi}{|\xi|} - w_{m'} \right| \ \text{ for every } w_{m'} \in M,\ m' \neq m \right\}.$

$\widehat{P_k g_m^j} = \chi_{\tau_m^{j,k}} \widehat{g}$.

$e^{\pm it\sqrt{-\Delta}} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i(x \cdot \xi \pm t|\xi|)} d\xi$.

Rescaled Lorentz Transformation:

$T_w^{2^j}(w,1) = (w,1)$,

$T_w^{2^j}(w,-1) = 2^{2j}(w,-1)$,

$T_w^{2^j}(x,t) = 2^j(x,t)$ if $(x,t) \in \mathbb{R}^{d+1}$ is orthogonal to $(w,1)$ and $(w,-1)$.

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